Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 31, 2004*
Accepted: *May 6, 2004*

$\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ supercoset and elliptic genera of Non-compact Calabi-Yau Manifolds

Tohru Eguchi and Yuji Sugawara

Department of Physics, Faculty of Science, University of Tokyo
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan
E-mail: eguchi@hep-th.phys.s.u-tokyo.ac.jp,
sugawara@hep-th.phys.s.u-tokyo.ac.jp

Abstract: We first discuss the relationship between the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ supercoset and $\mathcal{N}=2$ Liouville theory and make a precise correspondence between their representations. We shall show that the discrete unitary representations of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theory correspond exactly to those massless representations of $\mathcal{N}=2$ Liouville theory which are closed under modular transformations and studied in our previous work [18].
It is known that toroidal partition functions of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theory (2D Black Hole) contain two parts, continuous and discrete representations. The contribution of continuous representations is proportional to the space-time volume and is divergent in the infinite-volume limit while the part of discrete representations is volume-independent.
In order to see clearly the contribution of discrete representations we consider elliptic genus which projects out the contributions of continuous representations: making use of the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$, we compute elliptic genera for various non-compact space-times such as the conifold, ALE spaces, Calabi-Yau 3-folds with A_n singularities etc. We find that these elliptic genera in general have a complex modular property and are not Jacobi forms as opposed to the cases of compact Calabi-Yau manifolds.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Conformal Field Models in String Theory, Black Holes in String Theory.

http://jhep.sissa.it/archive/papers/jhep052004014/jhep052004014.pdf

Contents

1. Introduction

Study of superstring vacua in irrational superconformal theories has been a challenging problem. These theories describe superstrings propagating in non-compact curved space-time often developing isolated singularities. Applications of the study of this subject include the string theory on Calabi-Yau singularities and its holographic description, or the little string theory on NS5-branes in the T-dual picture [1]–[13]. The analysis is also important for backgrounds with non-trivial time dependence studied in the context of time-like Liouville theory [14, 15, 16]. Much of the efforts of understanding irrational (super)conformal theories have centered around the study of Liouville field theory. See, for instance, [17] for a recent review including a detailed list of literature.

One of the outstanding difficulties in these non-compact models is the coexistence of both discrete and continuous spectra of primary fields. Characters of these representations mix in a non-trivial manner under modular transformations in contrast with rational theories.

As an example, let us recall the superstring vacua of the type

$$\text{Minkowski space-time} \otimes \mathcal{N} = 2 \text{ minimal} \otimes \mathcal{N} = 2 \text{ Liouville}\,.$$

Then, we had the following puzzle;[1]

1. These backgrounds correspond to isolated singularities in Calabi-Yau spaces [1, 2, 3, 4], and one expects the existence of massless excitations describing the deformation of the singularities in the string spectra.

2. However, the modular invariant partition functions for such superconformal systems (in the infinite-volume limit) contain only contributions from continuous representations which possess the mass gap and no discrete representations corresponding to massless excitations occur, as was studied for instance in [7].

   In our previous work [18] we have partially resolved this puzzle by showing that

3. Discrete massless modes representing non-trivial cycles appear in the *open* string channels of cylinder amplitudes associated with supersymmetric boundary states. We have obtained candidate Cardy boundary states which correspond to ZZ and FZZT branes in the bosonic Liouville theory [19, 20, 21]. See also [22].

In this paper in order to further study these problems, we analyze the $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1)$ Kazama-Suzuki model [23], which is known to be T-dual (mirror) to the $\mathcal{N} = 2$ Liouville theory [24, 5, 25, 26]. After identifying the $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1)$ conformal blocks (branching functions) with the $\mathcal{N} = 2$ characters, we perform the character expansion of the toroidal partition function of the Kazama-Suzuki model in order to study its closed string spectrum. We will follow the analysis given in [27] (see also [28, 29]) of the partition function of the bosonic $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-model [30, 31]. It is well-known that the gauged WZW model for $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ describes the geometry of a 2-dimensional black hole [32].

We find an infra-red divergence in the partition function of the theory corresponding to the infinite volume V of the geometry of 2D black hole. In the limit $V \to \infty$ the partition function becomes simply the diagonal sum of continuous representations. When we suitably regulate the IR divergence, however, we find a non-trivial weight function for the continuous series and also a set of discrete representations with spin $1/2 \leq j \leq (k+1)/2$. We find that these discrete series are exactly the massless representations in $\mathcal{N} = 2$ Liouville theory which are closed under modular transformations [18].

Discrete states are localized around the tip of the cigar in 2D black hole [33]. Thus they are suppressed in the infinite-volume limit as compared with the continuous representations.

In order to see clearly identify the contributions of the discrete states not buried under the continuous representations, we propose to study the elliptic genus of the theory [34] to which the continuous representations do not contribute. We consider coupled superconformal systems: $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1) \otimes \mathcal{M}$, where $\mathcal{M}$ are various $\mathcal{N} = 2$ RCFT's. By taking suitable choices for $\mathcal{M}$ we have computed elliptic genera for the conifold, ALE spaces,

[1] This issue was first raised in [9].

JHEP05(2004)014

Calabi-Yau 3-folds with A_n singularities etc. It turns out that in general elliptic genera have a complicated modular property and are not Jacobi forms as in the case of compact Calabi-Yau manifolds. They are instead expressed in terms of the Appell function which features in the study of higher rank vector bundles over elliptic curves [35, 36].

While preparing this manuscript, we became aware of an interesting paper [37] on the e-print Archive, where the authors studied the modular properties of the extended characters in the bosonic $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-model defined in the similar manner as in [18], and there are some overlaps with the present work.

2. Toroidal partition function for the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model

2.1 Preliminaries

The Kazama-Suzuki supercoset model [23] for $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1)$ is defined as the coset CFT

$$\frac{\mathrm{SL}(2;\mathbb{R})_\kappa \times \mathrm{SO}(2)_1}{\mathrm{U}(1)_{-(\kappa-2)}}, \tag{2.1}$$

which is an $\mathcal{N}=2$ SCFT with the central charge and level[2]

$$\hat{c} \equiv \frac{c}{3} = 1 + \frac{2}{k}, \qquad k \equiv \kappa - 2. \tag{2.2}$$

More explicitly, the total world-sheet action is written as

$$\begin{aligned} S(g,A,\psi^\pm,\tilde{\psi}^\pm) &= \kappa S_{\mathrm{gWZW}}(g,A) + S_\psi(\psi^\pm,\tilde{\psi}^\pm,A), &(2.3)\\ \kappa S_{\mathrm{gWZW}}(g,A) &= \kappa S^{\mathrm{SL}(2;\mathbb{R})}_{\mathrm{WZW}}(g) + \frac{\kappa}{\pi}\int_\Sigma d^2z \Big\{ \mathrm{Tr}\left(\frac{\sigma_2}{2} g^{-1}\partial_{\bar z} g\right) A_z + \mathrm{Tr}\left(\frac{\sigma_2}{2}\partial_z g g^{-1}\right) A_{\bar z} + \\ &\qquad + \mathrm{Tr}\left(\frac{\sigma_2}{2} g \frac{\sigma_2}{2} g^{-1}\right) A_{\bar z} A_z + \frac{1}{2} A_{\bar z} A_z \Big\}, &(2.4)\\ S^{\mathrm{SL}(2;\mathbb{R})}_{\mathrm{WZW}}(g) &= -\frac{1}{8\pi}\int_\Sigma d^2z\ \mathrm{Tr}\left(\partial_\alpha g^{-1}\partial_\alpha g\right) + \frac{i}{12\pi}\int_B \mathrm{Tr}\left((g^{-1}dg)^3\right), &(2.5)\\ S_\psi(\psi^\pm,\tilde{\psi}^\pm,A) &= \frac{1}{2\pi}\int d^2z \Big\{ \psi^+(\partial_{\bar z} - A_{\bar z})\psi^- + \psi^-(\partial_{\bar z} + A_{\bar z})\psi^+ + \\ &\qquad + \tilde{\psi}^+(\partial_z - A_z)\tilde{\psi}^- + \tilde{\psi}^-(\partial_z + A_z)\tilde{\psi}^+ \Big\}, &(2.6) \end{aligned}$$

where the complex fermions $\psi^\pm$ (and $\tilde{\psi}^\pm$) have charge ± 1 with respect to the U(1)-gauge group. The bosonic part $\kappa S_{\mathrm{gWZW}}(g,A)$ is the gauged WZW model for the coset $\mathrm{SL}(2;\mathbb{R})_\kappa/\mathrm{U}(1)_A$ [38, 30], where $\mathrm{U}(1)_A$ indicates the gauging the axial U(1)-symmetry; $g \to \Omega g \Omega$, $\Omega(z,\bar z) = e^{iu(z,\bar z)\sigma_2}$ ($u(z,\bar z) \in \mathbb{R}$, σ_2 is the Pauli matrix.) It is well-known that this model describes string theory on 2D euclidean black-hole with the cigar geometry [32]. The WZW action $\kappa S^{\mathrm{SL}(2;\mathbb{R})}_{\mathrm{WZW}}(g)$ is formally equal to $-\kappa S^{\mathrm{SU}(2)}_{\mathrm{WZW}}(g)$, and has a negative signature in $i\sigma_2$-direction. Since we have $H^3(\mathrm{SL}(2;\mathbb{R})) = 0$, the action $\kappa S^{\mathrm{SL}(2;\mathbb{R})}_{\mathrm{WZW}}(g)$ can be rewritten as a purely two dimensional form and the level κ need not be an integer.

[2]Throughout this paper we denote the level of super $\mathrm{SL}(2;\mathbb{R})$ as k, and the level of bosonic $\mathrm{SL}(2;\mathbb{R})$ as $\kappa = k+2$.

The chiral currents

$$j^A(z) = \kappa\,\mathrm{Tr}\left(T^A\partial_z g g^{-1}\right)\,, \qquad \tilde{j}^A(\bar{z}) = -\kappa\,\mathrm{Tr}\left(T^A g^{-1}\partial_{\bar{z}} g\right)\,, \tag{2.7}$$

$$T^3 = \frac{1}{2}\sigma_2\,, \qquad T^{\pm} = \pm\frac{1}{2}\left(\sigma_3 \pm i\sigma_1\right) \tag{2.8}$$

satisfy the affine $\widehat{SL}(2;\mathbb{R})_\kappa$ current algebra (we write the left-mover only);

$$\begin{cases} j^3(z)j^3(0) \sim -\dfrac{\kappa/2}{z^2} \\ j^3(z)j^{\pm}(0) \sim \dfrac{\pm 1}{z}j^{\pm}(0) \\ j^+(z)j^-(0) \sim \dfrac{\kappa}{z^2} - \dfrac{2}{z}j^3(0) \end{cases} \tag{2.9}$$

and the pair of free fermions ψ^+, ψ^- satisfy the OPE's $\psi^+(z)\psi^-(0) \sim 1/z$, $\psi^{\pm}(z)\psi^{\pm}(0) \sim 0$. The explicit realization of $\mathcal{N}=2$ SCA is given by

$$\begin{aligned} &T(z) = \frac{1}{k}\eta_{AB}j^Aj^B + \frac{1}{k}J^3J^3 - \frac{1}{2}(\psi^+\partial\psi^- - \partial\psi^+\psi^-)\,, \qquad (\eta_{AB} = \mathrm{diag}(1,1,-1))\,, \\ &J = \psi^+\psi^- + \frac{2}{k}J^3\,, \qquad G^{\pm} = \frac{1}{\sqrt{k}}\psi^{\pm}j^{\mp}\,, \end{aligned} \tag{2.10}$$

where we set $J^3 \equiv j^3 + \psi^+\psi^-$, which is the unique U(1)-current commuting with all the generators of $\mathcal{N}=2$ SCA and hence defines the denominator of the $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1)$-supercoset.

To close this preliminary subsection, we summarize irreducible representations of $\widehat{\mathrm{SL}}(2;\mathbb{R})_\kappa$ current algebra. We concentrate on the representations with conformal weights bounded from below[3] and corresponding to unitary representations of the zero-mode subalgebra $\{j_0^3,\, j_0^{\pm}\}$. The parameter j is related to the conformal weight of vacuum states by the well-known formula

$$h = -\frac{j(j-1)}{\kappa-2}\,, \tag{2.11}$$

in all cases.

1. *continuous series:* $\hat{\mathcal{C}}_{p,\alpha}$ ($j = \frac{1}{2} + ip$, $p \in \mathbb{R}_{\geq 0}$, $0 \leq \alpha < 1$).

   They are non-degenerate representations and all the states lie above the mass gap $h \geq \frac{1}{4(\kappa-2)}$. The vacua have the j_0^3-spectrum; $j_0^3 = \alpha + n$, $n \in \mathbb{Z}$. The character formula is simply given by ($q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi i u}$);

$$\chi_{p,\alpha}(\tau,u) = \frac{q^{\frac{p^2}{\kappa-2}}}{\eta(\tau)^3}\sum_{n\in\mathbb{Z}} y^{n+\alpha}\,. \tag{2.12}$$

[3]We have more general representations constructed by the spectral flows;

$$j_m^3 \to j_m^3 - \frac{\kappa}{2}n\delta_{m,0}\,, \qquad j_m^{\pm} \to j_{m\pm n}^{\pm}\,, \qquad L_m \to L_m + nj_m^3 - \frac{\kappa}{4}n^2\delta_{m,0}\,.$$

However, they generically have unbounded spectra of conformal weights.

JHEP05(2004)014

In the following arguments we often use formal identities such as

$$\frac{1}{1-y}+\frac{y^{-1}}{1-y^{-1}}=\sum_{n\in\mathbb{Z}}y^n=\sum_{m\in\mathbb{Z}}\delta(u+m)\,,\qquad (z\in\mathbb{R}).$$

For a more rigorous treatment, one may consider the "regularized characters" $\chi_\xi(\tau,u;\epsilon)$ defined by replacing $e^{2\pi i(n+\alpha)u}$ with $e^{2\pi i(n+\alpha)u}e^{-|n|\epsilon}$ ($\epsilon>0$, $n\in\mathbb{Z}$, $0\le\alpha<1$).

2. *discrete series:* $\hat{\mathcal{D}}_j^{\pm}$ ($j\in\mathbb{R}_{>0}$).

   The superscript + indicates that these are the lowest weight representations and − does the highest weight ones. The spin parameter j is allowed to be continuous despite the name "discrete series" (while, in the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$-coset theory j takes discrete values). The vacua have the j_0^3-spectrum; $j_0^3=\pm(j+n)$, $n\in\mathbb{Z}_{\ge 0}$ for $\hat{\mathcal{D}}_j^{\pm}$ respectively. The character formula is written as

$$\chi_j^{\pm}(\tau,u)=\pm i\frac{q^{-\frac{1}{\kappa-2}\left(j-\frac{1}{2}\right)^2}y^{\pm\left(j-\frac{1}{2}\right)}}{\theta_1(\tau,u)}\,. \tag{2.13}$$

3. *identity representation.*

   This is the representation generated by the vacuum $h=j_0^3=0$ corresponding to the identity operator. Since this vacuum is both highest and lowest weight, the character formula becomes

$$\chi_0(\tau,u)=i\frac{q^{-\frac{1}{4(\kappa-2)}}y^{-1/2}(1-y)}{\theta_1(\tau,u)}\equiv\frac{q^{-\frac{\kappa}{8(\kappa-2)}}}{\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)}\,. \tag{2.14}$$

4. *complementary representations:* $\hat{\mathcal{E}}_{j,\alpha}$ ($0<j<\frac{1}{2}$, $\left|j-\frac{1}{2}\right|<\left|\alpha-\frac{1}{2}\right|$, $0\le\alpha<1$).

   These are non-degenerate representations below the mass gap. The vacua again have the j_0^3-spectrum; $j_0^3=\alpha+n$, ($n\in\mathbb{Z}$). The character formula has the same form as the continuous series. The range of j comes from the unitarity of zero-mode subalgebra. At the "boundary" of this range $j=0$, $\alpha=0$, the representation with $h=0$ becomes reducible and is decomposed as

$$\hat{\mathcal{E}}_{j=0,\alpha=0}\cong(\text{identity rep.})\oplus\hat{\mathcal{D}}^{+}_{j=1}\oplus\hat{\mathcal{D}}^{-}_{j=1}\,. \tag{2.15}$$

2.2 Branching functions

We start our analysis by identifying the conformal blocks for the toroidal partition function in the $\mathrm{SL}(2;\mathbb{R})_k/\,\mathrm{U}(1)$ Kazama-Suzuki model [23]. Although this task has been already implicitly carried out in [39],[4] it is helpful for our later analysis to provide the explicit formulas of conformal blocks. It turns out that these blocks are identified with the irreducible characters of $\mathcal{N}=2$ SCA. This fact will be the simplest evidence for T-duality with the $\mathcal{N}=2$ Liouville theory.

[4] It is an easy exercise to show the equivalence of the "non-compact parafermion" approach (with a compact boson added) used in [39] with the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki model. We can thus associate all the branching functions of the Kazama-Suzuki model with unitary irreducible representations of $\mathcal{N}=2$ SCA based on the analysis given in [39]. See also [40].

JHEP05(2004)014

We focus on the NS-sector and the formulas for other spin structures are obtained by using the 1/2-spectral flow. According to the standard treatment of coset CFT, the conformal blocks are defined as the following branching functions;

$$\chi_\xi(\tau,u)\frac{\theta_3(\tau,v)}{\eta(\tau)}=\sum_m \chi^{(\mathrm{NS})}_{\xi,m}(\tau,z)\frac{q^{-\frac{m^2}{k}}e^{2\pi imw}}{\eta(\tau)}\,. \qquad (2.16)$$

where ξ indicates each of the irreducible representations of $\widehat{\mathrm{SL}}(2;\mathbb{R})_\kappa$ classified above. The angle variables u, v, z, w are associated to the U(1)-currents j^3, $\psi^+\psi^-$, J, J^3 respectively, and we can easily read off the relations among them as

$$u=\frac{2}{k}z+w\,,\qquad v=u+z\equiv\frac{k+2}{k}z+w\,, \qquad (2.17)$$

from the definitions $J^3=j^3+\psi^+\psi^-$, $J=\psi^+\psi^-+\frac{2}{k}J^3$. The summation of m runs over the possible J^3_0-spectrum of representation ξ (tensored with the free fermion system $\psi^\pm$)

The branching functions for continuous series (and the complementary representations) are easily obtained due to the absence of null states;

$$\boxed{\chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{p,m}(\tau,z)=q^{\frac{p^2+m^2}{k}}e^{2\pi i\frac{2m}{k}z}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\equiv \mathrm{ch}^{(\mathrm{NS})}\left(h=\frac{p^2}{k}+\frac{1}{4k}+\frac{m^2}{k},Q=\frac{2m}{k};\tau,z\right)} \qquad (2.18)$$

Here $\mathrm{ch}^{(\mathrm{NS})}$ denotes the $\mathcal{N}=2$ irreducible character for the massive (non-degenerate) representation (B.1). We sometimes allow the pure imaginary values of p corresponding to the branching functions for the complementary representations $\hat{\mathcal{E}}_{j,\alpha}$ ($j=\frac{1}{2}+ip$). The unitarity condition [41] is derived from those for (the zero-mode parts of) $\hat{\mathcal{C}}_{p,\alpha}$, $\hat{\mathcal{E}}_{j=\frac{1}{2}+ip,\alpha}$;

$$p^2+\left(\alpha-\frac{1}{2}\right)^2\geq 0\,,\qquad \alpha\equiv m\ (\mathrm{mod}\,1)\,,\quad 0\leq\alpha<1\,. \qquad (2.19)$$

Derivation of the branching functions for the discrete series is more non-trivial. We will focus on $\hat{\mathcal{D}}^+_j$ because $\hat{\mathcal{D}}^-_j$ is obtained by the spectral flow

$$q^{-\frac{\kappa}{4}}e^{-2\pi i\frac{\kappa}{2}u}\chi^+_j(\tau,u+\tau)=\chi^-_{\frac{\kappa}{2}-j}(\tau,u)\,. \qquad (2.20)$$

The desired branching function is expressed as

$$\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j+n}(\tau,z)\equiv q^{\frac{(j+n)^2}{k}}\int_0^1 dw\,\chi^+_j(\tau,u)\theta_3(\tau,v)e^{-2\pi i(j+n)w}\,. \qquad (2.21)$$

We first consider the following shifts of angular variables in (2.21) (or (2.16));

$$z\longmapsto z+n\tau\,,\qquad w\longmapsto w-\frac{2}{k}n\tau\,,\qquad ({}^\forall n\in\mathbb{Z}) \qquad (2.22)$$

which leaves u invariant and causes $v\longmapsto v+n\tau$. Using the property $\theta_3(\tau,v+n\tau)=q^{-\frac{n^2}{2}}e^{-2\pi inv}\theta_3(\tau,v)$, we find the relation

$$q^{\frac{\hat{c}}{2}n^2}e^{2\pi i\hat{c}nz}\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j}(\tau,z+n\tau)=\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j+n}(\tau,z)\,,\qquad \left(\hat{c}=1+\frac{2}{k}\right). \qquad (2.23)$$

Therefore, it is enough to calculate $\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j}(\tau,z)$. The easiest way to evaluate it is to use of the character relation

$$\chi^{+}_{j}(\tau,u)+q^{\frac{\kappa}{4}-j}e^{2\pi i\frac{\kappa-2}{2}u}\chi^{-}_{\frac{\kappa}{2}-j}(\tau,u)=\chi_{j,\alpha=j}(\tau,u)\,. \tag{2.24}$$

r.h.s. is the character of complementary representation $\hat{\mathcal{E}}_{j,\alpha=j}$.

We act by $q^{\frac{j^2}{k}}\int_0^1 dw\theta_3(\tau,v)e^{-2\pi ijw}$ on both sides of (2.24). We also need the branching relation for $\hat{\mathcal{D}}^{-}_{\frac{\kappa}{2}-j}$;

$$\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j+n}(\tau,z)\;=\;q^{\frac{\left(j-\frac{k}{2}+n\right)^2}{k}}\int_0^1 dw\,\chi^{-}_{\frac{\kappa}{2}-j}(\tau,u)\theta_3(\tau,v)e^{-2\pi i\left(j-\frac{k}{2}+n\right)w}\,, \tag{2.25}$$

which is derived from (2.20) by making a shift $w\longmapsto w+\tau$ with keeping z (that is, $u\longmapsto u+\tau$, $v\longmapsto v+\tau$) in (2.21). We then obtain

$$(1+e^{2\pi iz}q^{1/2})\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j}(\tau,z)=\chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{j,m=j}(\tau,z)\equiv q^{\frac{j}{k}-\frac{1}{4k}}e^{2\pi i\frac{2j}{k}z}\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,. \tag{2.26}$$

This leads to the formula

$$\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j}(\tau,z)=\frac{q^{\frac{j}{k}-\frac{1}{4k}}e^{2\pi i\frac{2j}{k}z}}{1+e^{2\pi iz}q^{1/2}}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\equiv \mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}\left(Q=\frac{2j}{k};\tau,z\right)\,. \tag{2.27}$$

Here $\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q;\tau,z)$ is the massless matter character of $\mathcal{N}=2$ SCA for the chiral primary states $h=Q/2>0$ (B.2). Since (2.23) is the proper relation for spectral flow of $\mathcal{N}=2$ SCA (see (B.5)), we can identify $\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j+n}(\tau,z)$ with the flowed massless matter character (B.5);

$$\boxed{\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j+n}(\tau,z)=\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}\left(Q=\frac{2j}{k},n;\tau,z\right)\equiv\frac{q^{\frac{j+n^2+2nj}{k}-\frac{1}{4k}}e^{2\pi i\frac{2(j+n)}{k}z}}{1+e^{2\pi iz}q^{n+1/2}}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,,\quad({}^\forall n\in\mathbb{Z})} \tag{2.28}$$

One may check directly the validity of our branching relation

$$\chi^{+}_{j}(\tau,u)\frac{\theta_3(\tau,v)}{\eta(\tau)}=\sum_{n\in\mathbb{Z}}\frac{q^{\frac{j+n^2+2nj}{k}-\frac{1}{4k}}e^{2\pi i\frac{2(j+n)}{k}z}}{1+e^{2\pi iz}q^{n+1/2}}\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\cdot\frac{q^{-\frac{(j+n)^2}{k}}e^{2\pi i(j+n)w}}{\eta(\tau)}\,, \tag{2.29}$$

by comparing the residues at poles $e^{2\pi iu}q^m=1$ $(m\in\mathbb{Z})$ of both sides.

It is useful to note:

- $n\geq 0$: The vacuum state of $\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j+n}(\tau,z)$ is $(j^{+}_{0})^{n}|j,j\rangle\otimes|0\rangle_{\psi}$, which possesses the quantum numbers

$$h=\frac{2j\left(n+\frac{1}{2}\right)+n^2}{k}\,,\qquad Q=\frac{2(j+n)}{k}\,, \tag{2.30}$$

- $n<0$: The vacuum state is $(j^{-}_{-1})^{|n|-1}|j,j\rangle\otimes\psi^{-}_{-1/2}|0\rangle_{\psi}$, which has

$$h=\frac{-\left(k-2j\right)\left(n+\frac{1}{2}\right)+n^2}{k}\,,\qquad Q=\frac{2(j+n)}{k}-1\,. \tag{2.31}$$

Especially, $\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j,j-1}(\tau,z)$ is the character of anti-chiral primary with $h=-\frac{Q}{2}=\frac{1}{k}(\frac{k+2}{2}-j)$. As is proved in [39], the unitarity bound for the Casimir parameter j is given as

$$0<j<\frac{\kappa}{2}\left(\equiv\frac{k+2}{2}\right), \tag{2.32}$$

which reproduces all the (spectrally flowed) massless matter representations of $\mathcal{N}=2$ SCA lying on the "unitarity segments" given in [41].

Branching functions for the identity representation $\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{j=0,m=0}(\tau,z)$ may be derived in a similar manner with the help of (2.15). As one may expect, it is given by the graviton representation $h=Q=0$ (B.3) of $\mathcal{N}=2$ SCA. We find that

$$\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{0,0}(\tau,z)=\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(\tau,z)=q^{-\frac{1}{4k}}\frac{1-q}{(1+e^{2\pi iz}q^{1/2})(1+e^{-2\pi iz}q^{1/2})}\frac{\theta_3(\tau,z)}{\eta(\tau)^3} \tag{2.33}$$

Spectral-flowed version is given by

$$\boxed{\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{0,n}(\tau,z)=\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(n;\tau,z)\equiv q^{-\frac{1}{4k}}\frac{(1-q)q^{\frac{n^2}{k}+n-\frac{1}{2}}e^{2\pi i(\frac{2n}{k}+1)z}}{\left(1+e^{2\pi iz}q^{n+1/2}\right)\left(1+e^{2\pi iz}q^{n-1/2}\right)}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}} \tag{2.34}$$

The corresponding vacua are slightly non-trivial;

- $n=0$: The vacuum is $|0,0\rangle\otimes|0\rangle_{\psi}$ with $h=Q=0$.
- $n\geq 1$: The vacuum is $(j^+_{-1})^{n-1}|0,0\rangle\otimes\psi^+_{-1/2}|0\rangle_{\psi}$, which has the quantum numbers

$$h=\frac{n^2}{k}+n-\frac{1}{2}, \qquad Q=\frac{2n}{k}+1\,. \tag{2.35}$$

- $n\leq -1$: The vacuum is $(j^-_{-1})^{|n|-1}|0,0\rangle\otimes\psi^-_{-1/2}|0\rangle_{\psi}$, which has the quantum numbers

$$h=\frac{n^2}{k}-n-\frac{1}{2}, \qquad Q=\frac{2n}{k}-1\,. \tag{2.36}$$

We finally introduce the branching functions of other spin structures to fix the convention in this paper. Let $\chi^{(\mathrm{NS})}_{*,m}(\tau,z)$ be the abbreviated notations of the branching functions considered above ($m=J^3_0$). We define

$$\begin{aligned}
\chi^{(\widetilde{\mathrm{NS}})}_{*,m}(\tau,z) &\equiv e^{-i\pi\frac{2m}{k}}\,\chi^{(\mathrm{NS})}_{*,m}\left(\tau,z+\frac{1}{2}\right),\\
\chi^{(\mathrm{R})}_{*,m+\frac{1}{2}}(\tau,z) &\equiv q^{\frac{k+2}{8k}}\,e^{i\pi z\frac{k+2}{k}}\,\chi^{(\mathrm{NS})}_{*,m}\left(\tau,z+\frac{\tau}{2}\right),\\
\chi^{(\widetilde{\mathrm{R}})}_{*,m+\frac{1}{2}}(\tau,z) &\equiv e^{-i\pi\frac{2m}{k}}\,q^{\frac{k+2}{8k}}\,e^{i\pi z\frac{k+2}{k}}\,\chi^{(\mathrm{NS})}_{*,m}\left(\tau,z+\frac{\tau}{2}+\frac{1}{2}\right).
\end{aligned} \tag{2.37}$$

2.3 Toroidal partition function

Let us analyze the toroidal partition function of $\mathrm{SL}(2;\mathbb{R})_k / \mathrm{U}(1)$ Kazama-Suzuki model. It can be evaluated by the path-integral approach as described in [30, 31] for the bosonic 2D black-hole model [32]. Here we present only the result and leave the detailed calculations to appendix C. For the NS sector of the theory, we obtain

$$\begin{aligned} Z^{(\mathrm{NS})}(\tau) &= \int \mathcal{D}[g, A, \psi^{\pm}, \tilde{\psi}^{\pm}]\, e^{-\kappa S_{\mathrm{gWZW}}(g,A) - S_{\psi}(\psi^{\pm},\tilde{\psi}^{\pm},A)} \\ &= C \int_0^1 ds_1 \int_0^1 ds_2 \, \frac{|\theta_3(\tau, s_1\tau - s_2)|^2}{|\theta_1(\tau, s_1\tau - s_2)|^2} \sum_{w,m\in\mathbb{Z}} \exp\left(-\frac{\pi k}{\tau_2}|(w+s_1)\tau - (m+s_2)|^2\right), \end{aligned} \tag{2.38}$$

where C is a normalization constant to be fixed later. The partition functions for other spin structures are obtained by simply replacing $\theta_3(\tau, s_1\tau - s_2)$ with $\theta_{[\sigma]}(\tau, s_1\tau - s_2)$, defined as $\theta_{[\mathrm{NS}]} = \theta_3$, $\theta_{[\widetilde{\mathrm{NS}}]} = \theta_4$, $\theta_{[\mathrm{R}]} = \theta_2$ and $\theta_{[\widetilde{\mathrm{R}}]} = i\theta_1$. Assuming the diagonal modular invariant for spin structures, we obtain the partition function

$$Z(\tau) = \frac{1}{2}\sum_{\sigma} Z^{(\sigma)}(\tau)\,. \tag{2.39}$$

Here $u \equiv s_1\tau - s_2$ $(0 \le s_1, s_2 \le 1)$ is the modulus of gauge field A. One can view the sum over m, n as summing over the momentum and winding modes of a compact boson Y which parameterizes the 2-dimensional gauge field A. Y has a radius $\sqrt{2k}$ which is the size of the asymptotic circle far from the tip of the cigar. With the canonical normalization $Y(z)Y(0) \sim -\ln z$ for the field Y, total anomaly free current defining the BRST charge (see [38]) is given as

$$J^3_{\mathrm{tot}} \equiv j^3 + \psi^+\psi^- + \sqrt{\frac{k}{2}}\, i\partial Y\,, \qquad \tilde{J}^3_{\mathrm{tot}} \equiv \tilde{j}^3 + \tilde{\psi}^+\tilde{\psi}^- - \sqrt{\frac{k}{2}}\, i\bar{\partial} Y\,. \tag{2.40}$$

Of course, these currents have no singular OPE's with the $\mathcal{N} = 2$ SCA generators (2.10), assuring their BRST-invariance.

The partition function (2.39) is modular invariant in a formal sense since the modulus integral $\int ds^1 ds^2$ is logarithmically divergent due to the double pole of $1/|\theta_1(\tau, s_1\tau - s_2)|^2$. The appearance of divergence is not surprising since the target space has an infinite volume. The evaluation of modulus integral with a suitable IR cut-off in (2.38) is quite useful in determining the closed string spectrum, as shown in [27] for the bosonic $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ model. We turn to this analysis from now on.

2.4 Expansion into branching functions

We expand the toroidal partition function into branching functions of $\mathrm{SL}(2;\mathbb{R})_k / \mathrm{U}(1)$ following the procedure of [27]. (See also [28, 29].) Although our result will be quite similar to the bosonic case [27], we will present our analysis for the supersymmetric case for the sake of completeness.

JHEP05(2004)014

We start with the partition function of NS sector (2.38). Using the Poisson resummation formula, we can rewrite it as;

$$Z^{(\mathrm{NS})}(\tau) = C\int_0^1 ds_1 \int_0^1 ds_2 \sqrt{\frac{\tau_2}{k}} \sum_{w,n\in\mathbb{Z}} \frac{|\theta_3(\tau, s_1\tau - s_2)|^2}{|\theta_1(\tau, s_1\tau - s_2)|^2} \times$$
$$\times e^{-2\pi\tau_2\left(\frac{n^2}{2k}+\frac{k}{2}(s_1+w)^2\right)-2\pi i n((s_1+w)\tau_1 - s_2)} . \qquad (2.41)$$

The s_2-integral is easily performed since s_2 appears only linearly in the exponent. The q-expansion of the theta function terms is expressed as the trace over the Hilbert space of various oscillators. We introduce the oscillator levels N, $\tilde{N}$ and also the operators l, $\tilde{l}$ defined as

$$l \equiv \sharp\{j_n^+, \psi_r^+\} - \sharp\{j_n^-, \psi_r^-\}, \qquad \tilde{l} \equiv \sharp\{\tilde{j}_n^+, \tilde{\psi}_r^+\} - \sharp\{\tilde{j}_n^-, \tilde{\psi}_r^-\}. \qquad (2.42)$$

The relevant Hilbert space is

$$\mathcal{H}^{\pm} \equiv \left[\mathcal{F}^{\pm}_{\mathrm{SL}(2)} \otimes \mathcal{F}_{\psi} \otimes \mathcal{F}_{Y} \otimes \mathcal{F}_{\mathrm{gh}}\right]_L \otimes \left[\mathcal{F}^{\pm}_{\mathrm{SL}(2)} \otimes \mathcal{F}_{\psi} \otimes \mathcal{F}_{Y} \otimes \mathcal{F}_{\mathrm{gh}}\right]_R , \qquad (2.43)$$

where $\mathcal{F}_*$ denotes the Fock space of each sector. Especially, in the SL(2;$\mathbb{R}$)-sector, $\mathcal{F}^{+}_{\mathrm{SL}(2)}$ and $\mathcal{F}^{-}_{\mathrm{SL}(2)}$ means respectively the ones associated to the lowest and highest weight representations of zero-modes: namely we have

$$\mathrm{Tr}_{\mathcal{F}^{\pm}_{\mathrm{SL}(2)}}\left(q^{N-1/8}e^{2\pi i u l}\right) = \frac{\pm i e^{\mp i\pi u}}{\theta_1(\tau,u)} . \qquad (2.44)$$

We thus obtain

$$Z^{(\mathrm{NS})}(\tau) = C\int_0^1 ds_1 \sqrt{\frac{\tau_2}{k}} \sum_{w,n\in\mathbb{Z}} e^{-2\pi\tau_2\left(\frac{n^2}{2k}+\frac{kw^2}{2}+kws_1+\frac{k}{2}s_1^2\right)} \times$$
$$\times \mathrm{Tr}_{\mathcal{H}^+}\left(e^{-2\pi\tau_2\left(N+\tilde{N}+(l+\tilde{l}+1)s_1-\frac{1}{4}\right)+2\pi i\tau_1(N-\tilde{N}-nw)}\right), \qquad (2.45)$$

where the trace is constrained by the condition $l-\tilde{l}=n$ imposed by the s_2-integral.

The s_1-integral is non-trivial since s_1 appears quadratically in the exponent. Following [28, 27], we linearize it by means of the Fourier transformation;

$$\sqrt{k\tau_2}e^{-2\pi\tau_2\frac{k}{2}s_1^2} = \int_{-\infty}^{\infty} dc\, e^{-\frac{\pi}{k\tau_2}c^2 - 2\pi i c s_1} . \qquad (2.46)$$

The s_1-integral is then easy to compute and gives

$$\int_{-\infty}^{\infty} dc \int_0^1 ds_1\, e^{-\frac{\pi}{k\tau_2}c^2 - 2\pi\tau_2\left(\frac{n^2}{2k}+\frac{k}{2}w^2\right) - 2\pi s_1\left(ic+\tau_2(kw+l+\tilde{l}+1)\right)} =$$
$$= \int_{-\infty}^{\infty} dc\, \frac{-e^{-\frac{\pi}{k\tau_2}c^2 - 2\pi\tau_2\left(\frac{n^2}{2k}+\frac{k}{2}w^2\right)}}{2\pi\left(ic+\tau_2(kw+l+\tilde{l}+1)\right)} \left\{e^{-2\pi\left(ic+\tau_2(kw+l+\tilde{l}+1)\right)} - 1\right\} \qquad (2.47)$$
$$= -\frac{1}{2\pi}\int_{C_2} dp\, \frac{e^{-2\pi\tau_2\left(2\frac{p^2}{k}+\frac{n^2}{2k}+\frac{k}{2}(w+1)^2+l+\tilde{l}+1\right)}}{ip+\frac{1}{2}\left(k(w+1)+l+\tilde{l}+1\right)} + \frac{1}{2\pi}\int_{C_1} dp\, \frac{e^{-2\pi\tau_2\left(2\frac{p^2}{k}+\frac{n^2}{2k}+\frac{k}{2}w^2\right)}}{ip+\frac{1}{2}\left(kw+l+\tilde{l}+1\right)} .$$

In the last line we set $c = 2\tau_2 p - i\tau_2 k$ in the first term, and set $c = 2\tau_2 p$ in the second term. The integration contours are defined as $C_1 : \mathrm{Im}\, p = 0$, $C_2 : \mathrm{Im}\, p = k/2$.

To proceed further it is useful to make use of the spectral flow associated to the total currents (2.40), defined symbolically as[5]

$$\mathcal{U}_n \equiv e^{in\Phi(0,0)}\,, \quad i\partial\Phi \equiv J^3_{\mathrm{tot}}\,, \quad i\bar{\partial}\Phi \equiv \tilde{J}^3_{\mathrm{tot}}\,, \qquad (n \in \mathbb{Z})\,. \tag{2.48}$$

It is easy to see that $\mathcal{U}_1$ acts as

$$\begin{aligned} \mathcal{U}_1^{-1}\, N\, \mathcal{U}_1 &= N + l + \tfrac{1}{2}\,, & \mathcal{U}_1^{-1}\, \tilde{N}\, \mathcal{U}_1 &= \tilde{N} + \tilde{l} + \tfrac{1}{2}\,, \\ \mathcal{U}_1^{-1}\, l\, \mathcal{U}_1 &= l + 1\,, & \mathcal{U}_1^{-1}\, \tilde{l}\, \mathcal{U}_1 &= \tilde{l} + 1\,, \\ \mathcal{U}_1^{-1}\, w\, \mathcal{U}_1 &= w + 1\,, & \mathcal{U}_1^{-1}\, n\, \mathcal{U}_1 &= n\,, \end{aligned} \tag{2.49}$$

and maps the Fock space $\mathcal{H}^+$ to $\mathcal{H}^-$. We thus find that

$$\begin{aligned} &\sum_{w,n\in\mathbb{Z}} e^{-2\pi\tau_2\left(\frac{n^2}{2k}+\frac{k}{2}(w+1)^2\right)} \mathrm{Tr}_{\mathcal{H}^+}\left(\frac{e^{-2\pi\tau_2\left(N+\tilde{N}+l+\tilde{l}+1-\frac{1}{4}\right)+2\pi i\tau_1\left(N-\tilde{N}-nw\right)}}{ip+\frac{1}{2}\left(k(w+1)+l+\tilde{l}+1\right)}\right) = \\ &= \sum_{w,n\in\mathbb{Z}} e^{-2\pi\tau_2\left(\frac{n^2}{2k}+\frac{k}{2}w^2\right)} \mathrm{Tr}_{\mathcal{H}^-}\left(\frac{e^{-2\pi\tau_2\left(N+\tilde{N}-\frac{1}{4}\right)+2\pi i\tau_1\left(N-\tilde{N}-nw\right)}}{ip+\frac{1}{2}\left(kw+l+\tilde{l}-1\right)}\right). \end{aligned} \tag{2.50}$$

Substituting (2.47) and (2.50) to (2.45), we can show ($\hat{c} \equiv (k+2)/k$)

$$\begin{aligned} Z(\tau) = \frac{C}{2\pi k}\sum_{w,n\in\mathbb{Z}}\Bigg[&\int_{C_1} dp e^{-2\pi\tau_2\left(\frac{n^2}{2k}+\frac{kw^2}{2}+2\frac{p^2+1/4}{k}-\frac{\hat{c}}{4}\right)} \mathrm{Tr}_{\mathcal{H}^+}\left(\frac{e^{-2\pi\tau_2(N+\tilde{N})+2\pi i\tau_1(N-\tilde{N}-nw)}}{ip+\frac{1}{2}(kw+l+\tilde{l}+1)}\right) - \\ &-\int_{C_2} dp e^{-2\pi\tau_2\left(\frac{n^2}{2k}+\frac{kw^2}{2}+2\frac{p^2+1/4}{k}-\frac{\hat{c}}{4}\right)} \mathrm{Tr}_{\mathcal{H}^-}\left(\frac{e^{-2\pi\tau_2(N+\tilde{N})+2\pi i\tau_1(N-\tilde{N}-nw)}}{ip+\frac{1}{2}(kw+l+\tilde{l}-1)}\right)\Bigg]. \end{aligned} \tag{2.51}$$

As in [28, 27], let us perform the contour deformation; $C_2 \to C_1$, which picks up extra contributions from simple poles within the range $0 \le \mathrm{Im}\, p \le k/2$. The partition function is now divided into two parts;

$$Z^{(\mathrm{NS})}(\tau) = Z_{\mathbf{con}}{}^{(\mathrm{NS})}(\tau) + Z_{\mathbf{dis}}{}^{(\mathrm{NS})}(\tau)\,, \tag{2.52}$$

where the first term includes the p-integration on the real axis (C_1) and the second corresponds to the sum of pole residues.

The first term $Z_{\mathbf{con}}{}^{(\mathrm{NS})}(\tau)$ is rewritten as

$$\begin{aligned} Z_{\mathbf{con}}{}^{(\mathrm{NS})}(\tau) = \frac{C}{2\pi k}&\sum_{w,n\in\mathbb{Z}}\int_{-\infty}^{\infty} dp\, e^{-2\pi\tau_2\left(\frac{n^2}{2k}+\frac{kw^2}{2}+2\frac{p^2+1/4}{k}-\frac{\hat{c}}{4}\right)} \times \\ &\times\Bigg[\mathrm{Tr}_{\mathcal{H}^+}\left(\frac{e^{-2\pi\tau_2(N+\tilde{N})+2\pi i\tau_1(N-\tilde{N}-nw)}}{ip+\frac{1}{2}(kw+l+\tilde{l}+1)}\right) - \\ &\quad - \mathrm{Tr}_{\mathcal{H}^-}\left(\frac{e^{-2\pi\tau_2(N+\tilde{N})+2\pi i\tau_1(N-\tilde{N}-nw)}}{ip+\frac{1}{2}(kw+l+\tilde{l}-1)}\right)\Bigg] \end{aligned} \tag{2.53}$$

[5]This is different from the standard spectral flow of $\mathcal{N}=2$ SCA, defined in the same way referring to the $\mathcal{N}=2$ U(1)-currents (see (B.4)). We note that the operators $\mathcal{U}_n$ preserve the total current (2.40) and hence make the BRST-charge invariant. $\mathcal{U}_n$ also preserves the $\mathcal{N}=2$ SCA generators (2.10).

Since the factor $l+\tilde{l}$ appears only in the denominators, the traces are logarithmically divergent when one sums over the states of the form $(j_0^+\tilde{j}_0^+)^r|\psi\rangle$ $((j_0^-\tilde{j}_0^-)^r|\psi\rangle)$ for $\mathrm{Tr}_{\mathcal{H}^+}$ ($\mathrm{Tr}_{\mathcal{H}^-}$). This divergence comes from the pole $s_1=s_2=0$ in (2.38), that is, the infinite volume effect. Since the exponent $\frac{n^2}{2k}+\frac{kw^2}{2}+2\frac{p^2+1/4}{k}-\frac{\hat{c}}{4}$ is the correct weights for the continuous representations, it is natural to expect that $Z_{\mathbf{con}}{}^{(\mathrm{NS})}(\tau)$ can be expressed in a form

$$Z_{\mathbf{con}}{}^{(\mathrm{NS})}(\tau) = \frac{C}{k}\sum_{w,n\in\mathbb{Z}}\int_0^\infty dp\,\rho(p,w,n)\,\chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{p,m}(\tau,0)\chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{p,\tilde{m}}(\bar{\tau},0)\,, \tag{2.54}$$

$$m\equiv\frac{n-kw}{2}\,,\qquad \tilde{m}\equiv-\frac{n+kw}{2}\,, \tag{2.55}$$

with a suitable spectral density $\rho(p,w,n)$. Here $\chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{p,m}(\tau,0)$ is the branching function of continuous series (2.18) and is an irreducible massive character of $\mathcal{N}=2$ SCA. Although there appear some ambiguities in regulating the IR divergence, a candidate expression for $\rho(p,w,n)$ has been proposed in [28, 27].

$$\rho(p,w,n)=\frac{1}{2\pi}2\log\epsilon+\frac{1}{2\pi i}\frac{d}{2dp}\log\frac{\Gamma\left(-ip+\frac{1}{2}-m\right)\Gamma\left(-ip+\frac{1}{2}+\tilde{m}\right)}{\Gamma\left(+ip+\frac{1}{2}-m\right)\Gamma\left(+ip+\frac{1}{2}+\tilde{m}\right)}\,, \tag{2.56}$$

where $\epsilon>0$, $\epsilon\approx0$ is the IR cut-off. The first term in (2.56) is interpreted as the volume factor. The second term has a non-trivial momentum dependence and is related to the reflection amplitudes of Liouville theory as is discussed in [28].

On the other hand, the pole contributions yield the sum over the branching functions of discrete series (2.27) (and (2.28)). We take the identification $j=-ip+1/2$ for the spin parameter j so that we have $e^{-2\pi\tau_2\frac{p^2}{k}}=e^{2\pi\tau_2\frac{1}{k}(j-\frac{1}{2})^2}$ (with this identification relevant poles are located in the region $j\geq0$). The pole occurs in the 2nd term of (2.51) at

$$j=\frac{1}{2}(kw+l+\tilde{l})\left(=\frac{1}{2}(kw-n+2l)=\frac{1}{2}(kw+n+2\tilde{l})\right). \tag{2.57}$$

Since only the poles located on the interval between C_1 and C_2 can contribute, we must impose

$$\frac{1}{2}\leq j\leq\frac{k+1}{2}\left(=\frac{\kappa-1}{2}\right). \tag{2.58}$$

Note that this range (2.58) coincides with the one derived in the bosonic model [27] (with respect to κ) and is strictly smaller than the unitarity bound (2.32) [39] for generic values of k. This bound also coincides with the one required by the analysis of reflection coefficients (two point functions on sphere) [5], and also with that obtained from the no-ghost theorem for the parent $\mathrm{SL}(2;\mathbb{R})$ theory [42] (see also [43, 44, 45]).

We also would like to emphasize that the restricted range (2.58) agrees exactly with the range of massless matter representations $\mathrm{Ch}_{\mathbf{M}}^{(\mathrm{NS})}(r,s)$ of $\mathcal{N}=2$ Liouville theory discussed in [18]. In fact under the correspondence of notations

$$j=\frac{s}{2K}\,,\qquad k=\frac{N}{K} \tag{2.59}$$

the range $1/2 \le j \le (k+1)/2$ maps to

$$K \le s \le N + K \tag{2.60}$$

which is exactly the range of massless representations closed under modular transformations.

Recalling the branching relation (2.25), the desired character expansion is obtained as

$$Z_{\mathbf{dis}}{}^{(\mathrm{NS})}(\tau) = \frac{C}{k} \sum_{w,n\in\mathbb{Z}} \sum_{j\in\mathcal{J}_{w,n}} a(j) \chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{\frac{k}{2}-j, m+\frac{k}{2}}(\tau,0) \chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{\frac{k}{2}-j,\tilde{m}+\frac{k}{2}}(\bar{\tau},0)\,, \tag{2.61}$$

$$\mathcal{J}_{w,n} \equiv \left[\frac{1}{2}, \frac{k+1}{2}\right] \cap \left(\frac{kw-n}{2} + \mathbb{Z}\right), \tag{2.62}$$

$$a(j) \equiv \begin{cases} 1 & \left(\frac{1}{2} < j < \frac{k+1}{2}\right) \\ \frac{1}{2} & \left(j = \frac{1}{2},\, \frac{k+1}{2}\right) \end{cases} \tag{2.63}$$

where m and $\tilde{m}$ are defined as above (2.55). The factor $a(j)$ is necessary to give a correct weight to the poles on the boundary, $j = 1/2$ and $(k+1)/2$.[6] We choose the normalization constant $C = k$.

We finally make a comment with respect to the modular invariance. The regularized partition function is written as

$$\begin{aligned} Z(\tau;\epsilon) = \frac{1}{2}\sum_{\sigma}\sum_{w,n}\Bigg[&\int_0^\infty dp\, \rho(p,w,n;\epsilon) \chi_{\mathbf{con}}{}^{(\sigma)}_{p,m}(\tau,0) \chi_{\mathbf{con}}{}^{(\sigma)}_{p,\tilde{m}}(\bar{\tau},0) + \\ &+ \sum_{j\in\mathcal{J}_{w,n}} a(j) \chi_{\mathbf{dis}}{}^{(\sigma)}_{\frac{k}{2}-j,m+\frac{k}{2}}(\tau,0) \chi_{\mathbf{dis}}{}^{(\sigma)}_{\frac{k}{2}-j,\tilde{m}+\frac{k}{2}}(\bar{\tau},0)\Bigg]\,, \end{aligned} \tag{2.64}$$

where we have indicated the dependence on IR cut-off ϵ explicitly. Strictly speaking this expression is *not* modular invariant because of the non-trivial p-dependence of the spectral density $\rho(p,w,n;\epsilon)$ (2.56), even though the original formula coming from the path-integral (2.38) appears modular invariant. In fact, the IR regularization would spoil the modular invariance, as we often face in general non-compact curved backgrounds. In order

[6] One may object to a fractional factor 1/2 in the weight of discrete representations in the partition function. However, there exists a character identity

$$\begin{aligned} \chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{\frac{1}{2},\frac{1}{2}+n}(\tau,z) + \chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{\frac{k+1}{2},\frac{k+1}{2}+n}(\tau,z) &= \chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{j=\frac{1}{2},m=\frac{1}{2}+n}(\tau,z) \\ &\equiv \mathrm{ch}^{(\mathrm{NS})}\left(h = \frac{1}{2k} + \frac{n+n^2}{k}, Q = \frac{2n+1}{k}; \tau, z\right). \end{aligned}$$

which enables us to write (2.61) in a form (set $C = k$)

$$Z_{\mathbf{dis}}{}^{(\mathrm{NS})}(\tau) = \sum_{w,n\in\mathbb{Z}} \sum_{j\in\mathcal{J}'_{w,n}} \chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{\frac{k}{2}-j,m+\frac{k}{2}}(\tau,0) \chi_{\mathbf{dis}}{}^{(\mathrm{NS})}_{\frac{k}{2}-j,\tilde{m}+\frac{k}{2}}(\bar{\tau},0) + (\text{terms including continuous rep.})$$

$$\mathcal{J}'_{w,n} \equiv \left(\frac{1}{2}, \frac{k+1}{2}\right] \cap \left(\frac{kw-n}{2} + \mathbb{Z}\right).$$

JHEP05(2004)014

to recover invariance, the best one can do is to consider the partition function *per unit volume*;

$$Z(\tau) \equiv \lim_{\epsilon\to+0} \frac{Z(\tau;\epsilon)}{2\frac{1}{2\pi}\log\epsilon} = \frac{1}{2}\sum_{\sigma}\sum_{w,n}\int_0^\infty dp\, \chi_{\mathbf{con}}{}^{(\sigma)}_{p,m}(\tau,0)\chi_{\mathbf{con}}{}^{(\sigma)}_{p,\tilde{m}}(\bar{\tau},0)\,. \tag{2.65}$$

The modular invariance of (2.65) follows from that of a free compact boson with a radius $R = \sqrt{2k}$.

To summarize, the partition function is decomposed into two parts: (1) the continuous part $Z_{\mathbf{con}}(\tau)$, and (2) the discrete part $Z_{\mathbf{dis}}(\tau)$. The continuous part $Z_{\mathbf{con}}(\tau)$ includes dominant contributions proportional to the volume factor, and correspond to the modes freely propagating in the bulk. Its precise definition depends on the regularization scheme.

On the other hand, the discrete part $Z_{\mathbf{dis}}(\tau)$ only contains representations of (anti-)chiral primaries and their spectral flows within the range (2.58). They describe excitations localized around the tip of 2D black-hole that could be identified as the bound states [33] (see also [46]). The absence of IR divergence in $Z_{\mathbf{dis}}(\tau)$ is in accord with this expectation. The part of discrete representations is universal: insensitive to the choice of regularization scheme and stable under marginal deformations preserving $\mathcal{N} = 2$ SUSY. We will make use of this piece to compute elliptic genus in the next section which captures the geometrical information of the singular space-time.

It seems that a strictly modular invariant partition function is obtained only after dividing by the infinite volume factor, which inevitably contains only the continuous representations. All the states appearing in this partition function (2.64) lie above the mass gap $h \geq 1/(4k)$, which corresponds to the decoupling of gravity in such a space-time.

3. Coupling to RCFT's

In this section we investigate the type-II string vacua of the forms;

$$\mathbb{R}^{d-1,1} \otimes \mathcal{M} \otimes \mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)\,, \tag{3.1}$$

where $\mathcal{M}$ is an arbitrary rational $\mathcal{N} = 2$ SCFT with $\hat{c} = \hat{c}_{\mathcal{M}}$. The criticality condition is

$$\frac{d}{2} + \hat{c}_{\mathcal{M}} + \frac{k+2}{k} = 5\,, \tag{3.2}$$

and we assume d is even. We expect that the superconformal system $\mathcal{M} \otimes \mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ describes a non-compact $CY_{\mathbf{n}}$ with $\mathbf{n} = 5 - d/2$. We assume a modular invariant of $\mathcal{M}$-sector with conformal blocks F_I as

$$Z_{\mathcal{M}}(\tau,z) = \frac{1}{2}\sum_{\sigma} Z^{(\sigma)}_{\mathcal{M}}(\tau,z) \equiv e^{-2\pi\hat{c}_{\mathcal{M}}\frac{(\mathrm{Im}\,z)^2}{\tau_2}}\frac{1}{2}\sum_{\sigma}\sum_{I,\tilde{I}} N_{I,\tilde{I}}F^{(\sigma)}_I(\tau,z)\tilde{F}^{(\sigma)}_{\tilde{I}}(\bar{\tau},\bar{z})\,, \tag{3.3}$$

where z is the angle associated to the U(1)-charge of $\mathcal{N} = 2$ SCA and σ denotes the spin structures as before. In this paper we use the conventions;

$$\begin{aligned} F_I^{(\widetilde{\rm NS})}(\tau,z) &\equiv e^{-i\pi Q(I)} F_I^{(\rm NS)}\left(\tau, z+\frac{1}{2}\right), \\ F_I^{(\rm R)}(\tau,z) &\equiv q^{\frac{\hat{c}_{\mathcal{M}}}{8}} e^{i\pi \hat{c}_{\mathcal{M}} z} F_I^{(\rm NS)}\left(\tau, z+\frac{\tau}{2}\right), \\ F_I^{(\widetilde{\rm R})}(\tau,z) &\equiv e^{-i\pi Q(I)} q^{\frac{\hat{c}_{\mathcal{M}}}{8}} e^{i\pi \hat{c}_{\mathcal{M}} z} F_I^{(\rm NS)}\left(\tau, z+\frac{\tau}{2}+\frac{1}{2}\right), \end{aligned} \tag{3.4}$$

where $Q(I)$ is the U(1)-charge of vacuum state of the conformal block $F_I^{(\rm NS)}(\tau,z)$. (Note that the U(1)-charge for $F_I^{(\rm R)}(\tau,z)$ is equal $Q(I)+\hat{c}_{\mathcal{M}}/2$.) Because of the rationality of $\mathcal{M}$ there exists a finite periodicity in integral spectral flows. We assume $N_0 \in \mathbb{Z}_{>0}$ to be the minimal integer such that

$$q^{\frac{\hat{c}_{\mathcal{M}}}{2}m^2} e^{2\pi i \hat{c}_{\mathcal{M}} m z} F_I^{(\rm NS)}(\tau, z+m\tau+n) = F_I^{(\rm NS)}(\tau,z)\,, \qquad ({}^{\forall}I\,,\ {}^{\forall}m, {}^{\forall}n \in N_0\mathbb{Z})\,. \tag{3.5}$$

Then we have $\hat{c}_{\mathcal{M}} = M/N_0$ with some positive integer M. Recalling the criticality condition (3.2), we find that

$$k=\frac{N}{K}\,, \qquad N=N_0 \text{ or } 2N_0\,, \qquad \left(\hat{c}_{\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)} = 1+\frac{2K}{N}\,,\ 1+\frac{2K}{N}+\frac{M}{N_0} = \mathbf{n}\right), \tag{3.6}$$

with some positive integer K.[7] Throughout this section we shall assume (3.6) with fixed positive integers N, K in the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-sector.

A typical example is the Gepner model [47];

$$\mathcal{M} = M_{n_1}\otimes\cdots\otimes M_{n_r}\,, \qquad \hat{c}_{\mathcal{M}} = \sum_{i=1}^{r}\frac{n_i}{n_i+2}\,, \tag{3.7}$$

where M_n is the level n $\mathcal{N}=2$ minimal model. The relevant conformal blocks are the products of minimal characters (A.6)

$$F_I^{(\rm NS)}(\tau,z) = \prod_{i=1}^{r} \mathrm{ch}_{\ell_i,m_i}^{(\rm NS)}(\tau,z)\,, \qquad (I\equiv((\ell_1,m_1),\ldots,(\ell_r,m_r)))\,, \tag{3.8}$$

and clearly we have $N_0 = \mathrm{L.C.M}\{n_i+2\}$.

We also assume the symmetry under spectral flow of the coefficients of the modular invariant $N_{I,\tilde{I}}$;

$$N_{s(I),s(\tilde{I})} = N_{I,\tilde{I}}\,, \tag{3.9}$$

where $s: I \to s(I)$ denotes the action of spectral flow

$$F_{s(I)}^{(\rm NS)}(\tau,z) = q^{\frac{\hat{c}_{\mathcal{M}}}{2}} e^{2\pi i \hat{c}_{\mathcal{M}} z} F_I^{(\rm NS)}(\tau, z+\tau)\,. \tag{3.10}$$

[7] Note that both of the pairs N_0, M and N, K are *not* necessarily relatively prime. For example, in the case

$$\mathbb{R}^{3,1}\otimes M_{2n}\otimes \mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)\,, \qquad (n\in\mathbb{Z}_{\geq 0})$$

we find $N_0=N=2n+2$, $M=2n$, $K=n+2$. Therefore, N_0, M are not relatively prime for any n, and N, K are also not for even n.

3.1 Modular invariant partition functions per unit volume

We first consider the modular invariant partition function for the coupled system $\mathcal{M} \otimes \mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ with $\hat{c} = \mathbf{n}$. We assume N and K are relatively prime for the time being. Let us recall the partition function of the $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1)$-sector defined by dividing by the volume factor (2.65). Now in the case of a rational level $k = N/K$, the partition function (2.65) can be rewritten in terms of the extended characters

$$Z(\tau) = \frac{1}{2}\sum_{\sigma}\sum_{w_0\in\mathbb{Z}_{2K}}\sum_{n_0\in\mathbb{Z}_N}\int_0^\infty dp\,\chi_{\mathbf{con}}{}^{(\sigma)}(p, Kn_0 - Nw_0;\tau,0)\times \times\chi_{\mathbf{con}}{}^{(\sigma)}(p,-Kn_0-Nw_0;\bar{\tau},0)\,, \qquad (3.11)$$

$$\boxed{\chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p,m;\tau,z) \equiv \sum_{n\in N\mathbb{Z}}\chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{p,\frac{m}{2K}+n}(\tau,z) \equiv q^{\frac{Kp^2}{N}}\Theta_{m,NK}\left(\tau,\frac{2z}{N}\right)\frac{\theta_3(\tau,z)}{\eta(\tau)^3}} \qquad (3.12)$$

This function is identified with the extended massive character (B.6) introduced in [18];

$$\boxed{\chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p,m;\tau,z) = \mathrm{Ch}^{(\mathrm{NS})}\left(h = \frac{Kp^2}{N} + \frac{m^2+K^2}{4NK}, Q = \frac{m}{N};\tau,z\right)} \qquad (3.13)$$

(3.11) is derived from the identity

$$\sum_{w,n\in\mathbb{Z}} q^{\frac{1}{4}\left(\sqrt{\frac{K}{N}}n-\sqrt{\frac{N}{K}}w\right)^2}\bar{q}^{\frac{1}{4}\left(\sqrt{\frac{K}{N}}n+\sqrt{\frac{N}{K}}w\right)^2} = = \sum_{w_0\in\mathbb{Z}_{2K}}\sum_{n_0\in\mathbb{Z}_N}\Theta_{Kn_0-Nw_0,NK}(\tau,0)\Theta_{-Kn_0-Nw_0,NK}(\bar{\tau},0)\,, \qquad (3.14)$$

and the modular invariant has the same form as the level NK theta system given in [48]. $\mathbb{Z}_N$-periodicity under integral spectral flows is easy to see;

$$q^{\frac{\hat{c}}{2}r^2}e^{2\pi i\hat{c}rz}\chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p,m;\tau,z+r\tau+n) = \chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p,m;\tau,z)\,, \quad {}^\forall r, {}^\forall n\in N\mathbb{Z}\,, \quad \left(\hat{c} = 1+\frac{2K}{N}\right). \qquad (3.15)$$

$\chi_{\mathbf{con}}{}^{(\sigma)}(p,m;\tau,z)$ for other spin structures are defined by the 1/2-spectral flows in the same way as (2.37).

Now, the task we have to carry out is the chiral projection onto integral U(1)-charges as in the Gepner models [47] while taking account of the twisted sectors generated by integral spectral flows (see [49]). Because of the periodicities (3.5), (3.15), this is reduced to a $\mathbb{Z}_N$-orbifoldization. The desired conformal blocks are thus defined as the flow invariant orbits [49][8]

$$\begin{aligned}
\mathcal{F}^{(\mathrm{NS})}_{I,p,w_0}(\tau,z) &= \frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i\mathbf{n}za}\, F^{(\mathrm{NS})}_I(\tau,z+a\tau+b)\times \\
&\qquad\qquad \times \chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p,Kn_I-Nw_0;\tau,z+a\tau+b)\,, \\
\tilde{\mathcal{F}}^{(\mathrm{NS})}_{\tilde{I},p,w_0}(\bar\tau,\bar z) &= \frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i\mathbf{n}\bar za}\, \tilde F^{(\mathrm{NS})}_{\tilde I}(\bar\tau,\bar z+a\bar\tau+b)\times \\
&\qquad\qquad \times \chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p,-Kn_I-Nw_0;\bar\tau,-\bar z-a\bar\tau-b)\,,
\end{aligned} \tag{3.16}$$

where $n_I\in\mathbb{Z}_N$ is the solution of the condition

$$\frac{Kn_I}{N}+Q(I)\in\mathbb{Z}\,,\qquad \frac{Kn_I}{N}+Q(\tilde I)\in\mathbb{Z}\,,\qquad N_{I,\tilde I}\neq 0\,, \tag{3.17}$$

which uniquely exist for each I, $\tilde I$ with $N_{I,\tilde I}\neq 0$ such that $Q(I)-Q(\tilde I)\in\mathbb{Z}$, since we have $Q(I),Q(\tilde I)\in\frac{1}{N}\mathbb{Z}$, $({}^\forall I,{}^\forall\tilde I)$ and N and K are assumed to be relatively prime. Especially, the solutions of (3.17) always exist if we assume the diagonal modular invariant in the $\mathcal{M}$-sector. We define $\mathcal{F}^{(\mathrm{NS})}_{I,p,w_0}\equiv 0$ if the solution n_I of (3.17) does not exist. The conformal blocks for other spin structures are defined by the 1/2-spectral flows[9]

$$\begin{aligned}
\mathcal{F}^{(\widetilde{\mathrm{NS}})}_{I,p,w_0}(\tau,z) &\equiv \mathcal{F}^{(\mathrm{NS})}_{I,p,w_0}\left(\tau,z+\frac{1}{2}\right)\,, \\
\mathcal{F}^{(\mathrm{R})}_{I,p,w_0}(\tau,z) &\equiv q^{\frac{\mathbf{n}}{8}} e^{2\pi i\frac{\mathbf{n}}{2}z}\,\mathcal{F}^{(\mathrm{NS})}_{I,p,w_0}\left(\tau,z+\frac{\tau}{2}\right)\,, \\
\mathcal{F}^{(\widetilde{\mathrm{R}})}_{I,p,w_0}(\tau,z) &\equiv q^{\frac{\mathbf{n}}{8}} e^{2\pi i\frac{\mathbf{n}}{2}z}\,\mathcal{F}^{(\mathrm{NS})}_{I,p,w_0}\left(\tau,z+\frac{\tau}{2}+\frac{1}{2}\right)\,.
\end{aligned} \tag{3.18}$$

Let us next consider the cases when N and K are not relatively prime. We set

$$\mathrm{G.C.D}\,\{N,\,K\}=\nu\,,\qquad N=\bar N\nu\,,\qquad K=\bar K\nu\,. \tag{3.19}$$

Then the solutions of the condition (3.17) exist only if $Q(I),Q(\tilde I)\in\frac{\nu}{N}\mathbb{Z}$, and are not unique: we must sum over the mod $\bar N$ spectral flows $n_I+2\bar N\mu$ ($\mu\in\mathbb{Z}_\nu$). Namely, (3.16) has to be replaced with

$$\begin{aligned}
\mathcal{F}^{(\mathrm{NS})}_{I,p,w_0}(\tau,z) &= \frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i\mathbf{n}za}\, F^{(\mathrm{NS})}_I(\tau,z+a\tau+b)\times \\
&\qquad \times \sum_{\mu\in\mathbb{Z}_\nu} \chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{(N,K)}(p,K(n_I+2\bar N\mu)-Nw_0;\tau,z+a\tau+b)
\end{aligned}$$

[8]In our convention the $\mathcal{N}=2$ U(1)-current in right-mover has been defined to be

$$\tilde J=-\tilde\psi^+\tilde\psi^- -\frac{2}{k}\tilde J^3\,,$$

and the minus sign of $\bar z$ in (3.16) reflects this fact. This convention is natural because the winding/KK-momentum w, n of the compact boson Y are correctly reinterpreted as those of the $\mathcal{N}=2$ U(1)-currents J, $\tilde J$ by using the BRST-invariance. (See (2.40).)

[9]We here adopt a somewhat unusual definitions of $\widetilde{\mathrm{NS}}$, $\widetilde{\mathrm{R}}$-conformal blocks omitting some phase factors. As an advantage, the supersymmetric conformal blocks (3.24) become simpler forms.

$$= \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} z a}\, F_I^{(\mathrm{NS})}(\tau, z + a\tau + b) \times$$

$$\times\, \chi_{\mathbf{con}}{}^{(\mathrm{NS})}_{(\bar{N},\bar{K})}(p, \bar{K} n_I - \bar{N} w_0; \tau, z + a\tau + b)\,, \tag{3.20}$$

where we have indicated explicitly the N, K dependence of the extended characters.

The modular invariant partition function (as the σ-model on $CY_{\mathbf{n}}$) is obtained as

$$Z(\tau, z) = e^{-2\pi \mathbf{n} \frac{(\mathrm{Im}\, z)^2}{\tau_2}} \frac{1}{2} \sum_{\sigma} \sum_{w_0 \in \mathbb{Z}_{2K}} \frac{1}{N} \sum_{I,\tilde{I}} \int_0^\infty dp\, N_{I,\tilde{I}} \mathcal{F}^{(\sigma)}_{I,p,w_0}(\tau, z) \mathcal{F}^{(\sigma)}_{\tilde{I},p,w_0}(\bar{\tau}, \bar{z})\,. \tag{3.21}$$

We note the invariance (up to phase) under spectral flow of the conformal blocks $\mathcal{F}^{(\sigma)}_{I,p,w_0}$

$$q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a z}\, \mathcal{F}^{(\sigma)}_{I,p,w_0}(\tau, z) = \epsilon_{a,b}(\sigma) \mathcal{F}^{(\sigma)}_{I,p,w_0}(\tau, z + a\tau + b)\,, \qquad ({}^\forall a, b \in \mathbb{Z})\,, \tag{3.22}$$

$$\epsilon_{a,b}(\mathrm{NS}) = 1\,, \quad \epsilon_{a,b}(\widetilde{\mathrm{NS}}) = (-1)^{\mathbf{n}a}\,, \quad \epsilon_{a,b}(\mathrm{R}) = (-1)^{\mathbf{n}b}\,, \quad \epsilon_{a,b}(\widetilde{\mathrm{R}}) = (-1)^{\mathbf{n}(a+b)}\,, \tag{3.23}$$

and recall the assumption (3.9). Then the factor $1/N$ in (3.21) is necessary to remove the N-fold overcounting of states.

Incorporating the $\mathbb{R}^{d-1,1}$-sector ($\frac{d}{2} + \mathbf{n} = 5$), the supersymmetric conformal blocks are constructed as

$$\frac{1}{2} \frac{1}{\tau_2^{\frac{d-2}{4}}\, \eta(\tau)^{d-2}} \sum_{\sigma} \epsilon(\sigma) \left(\frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)}\right)^{\frac{d-2}{2}} \left(\mathcal{F}^{(\sigma)}_{I,p,w_0}(\tau, z) + \mathcal{F}^{(\sigma)}_{I,p,w_0}(\tau, -z)\right), \tag{3.24}$$

where $\theta_{[\sigma]}$ again denotes θ_3, θ_4, θ_2, $i\theta_1$ for $\sigma = \mathrm{NS}, \widetilde{\mathrm{NS}}, \mathrm{R}, \widetilde{\mathrm{R}}$ respectively, and we set $\epsilon(\mathrm{NS}) = \epsilon(\widetilde{\mathrm{R}}) = +1$, $\epsilon(\widetilde{\mathrm{NS}}) = \epsilon(\mathrm{R}) = -1$. The conformal blocks (3.24) actually vanish for arbitrary τ, z [50], as is consistent with the space-time SUSY. It is not difficult to confirm that the conformal blocks (3.24) reproduce the results obtained in [7] for the special cases $\mathcal{M} = M_{n-2}$. (Precisely speaking, in the $d = 4$ case we need some further orbifoldization in the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-sector to reproduce the formula of [7].)

3.2 Elliptic genera

Let us next study the discrete spectrum of the theory which carries geometrical information of the target space geometry. As we have seen in the previous section, contributions of continuous representations dominate the partition functions and it is difficult to isolate the contributions of discrete states of the theory by inspecting the partition functions. We thus propose to study the elliptic genera from which continuous series decouple and one can clearly see the contents of discrete states in the theory.

We first recall that the elliptic genera are defined by the partition functions in the $\tilde{R}$ sector of the theory [34, 51],

$$\mathcal{Z}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_L^{(\mathrm{R})} \otimes \mathcal{H}_R^{(\mathrm{R})}} (-1)^F e^{2\pi i z J_0} q^{L_0 - \frac{\hat{c}}{8}} \bar{q}^{\tilde{L}_0 - \frac{\hat{c}}{8}}\,, \tag{3.25}$$

where $F \equiv F_L - F_R$ denotes the world-sheet fermion number. When one sets $z = 0$ above, elliptic genus is reduced to the Witten index.

It is well-known that, in any rational $\mathcal{N}=2$ SCFT the elliptic genus is a good supersymmetric index stable under arbitrary chiral marginal deformations. Furthermore, it possesses simple modular and spectral flow properties. It is identified as a (weak) Jacobi form [52] in mathematical terminology (see, e.g. [34, 53]).

As we shall see in the following, in the case of singular non-compact manifolds elliptic genera are no longer Jacobi forms and have some complicated modular properties: they are in general described by Appell functions which feature in the study of vector bundles of higher rank over elliptic curves [35, 36].

The evaluation of elliptic genus is almost parallel to the previous analysis of the modular invariant partition functions: we just replace the continuous extended characters ${\chi_{\mathbf{con}}}^{(\sigma)}(p,m;\tau,z)$ (3.12) by the discrete ones ${\chi_{\mathbf{dis}}}^{(\sigma)}(s,m;\tau,z)$ $(m\in\mathbb{Z}_{2NK})$ defined by

$$\boxed{\begin{array}{ll} {\chi_{\mathbf{dis}}}^{(\mathrm{NS})}(s,s+2Kr;\tau,z)\equiv \displaystyle\sum_{n\in N\mathbb{Z}} {\chi_{\mathbf{dis}}}^{(\mathrm{NS})}_{\frac{s}{2K},\frac{s}{2K}+r+n}(\tau,z)\equiv \mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(r,s;\tau,z) & (r\in\mathbb{Z}_N)\\ {\chi_{\mathbf{dis}}}^{(\mathrm{NS})}(s,m;\tau,z)\equiv 0\,, \qquad m\not\equiv s \ (\mathrm{mod}\, 2K) & \end{array}} \tag{3.26}$$

In this definition ${\chi_{\mathbf{dis}}}^{(\mathrm{NS})}_{j,m}(\tau,z)$ are the branching functions for discrete series (2.28) and identified with the $\mathcal{N}=2$ massless matter characters. $\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(r,s;\tau,z)$ is the massless extended character introduced in [18], given explicitly in (B.7). The extended characters of other spin structures are again defined by the spectral flows. We note that ${\chi_{\mathbf{dis}}}^{(\mathrm{R})}(s,m)$ (and ${\chi_{\mathbf{dis}}}^{(\widetilde{\mathrm{R}})}(s,m)$) can take non-zero values only if $m\equiv s+K \ (\mathrm{mod}\,2K)$. The discrete part of partition function (2.61) can be rewritten in terms of the extended characters ${\chi_{\mathbf{dis}}}^{(\sigma)}(s,m;\tau,z)$ in the same way as (3.11);

$$\begin{aligned} Z_{\mathbf{dis}}(\tau) = \frac{1}{2}\sum_{\sigma}\sum_{w_0\in\mathbb{Z}_{2K}}\sum_{n_0\in\mathbb{Z}_N}\sum_{s=K}^{N+K} a(s){\chi_{\mathbf{dis}}}^{(\sigma)}(s,Kn_0-Nw_0;\tau,0)\times \\ \times {\chi_{\mathbf{dis}}}^{(\sigma)}(s,-Kn_0-Nw_0;\bar{\tau},0)\,, \end{aligned} \tag{3.27}$$

$$a(s)\equiv\begin{cases} 1 & K+1\le s\le N+K-1 \\ \frac{1}{2} & s=K,\ N+K\,. \end{cases} \tag{3.28}$$

In the following let us assume that K and N are relatively prime for the sake of simplicity. Performing the $\mathbb{Z}_N$-orbifoldization, we can construct the conformal blocks in the same way as (3.16);

$$\begin{aligned} \mathcal{G}^{(\mathrm{NS})}_{I,s,w_0}(\tau,z) &= \frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{\mathbf{n}}{2}a^2}e^{2\pi i\mathbf{n}za}\,F^{(\mathrm{NS})}_I(\tau,z+a\tau+b)\times \\ &\qquad\times {\chi_{\mathbf{dis}}}^{(\mathrm{NS})}_{(N,K)}(s,Kn_I-Nw_0;\tau,z+a\tau+b)\,, \\ \tilde{\mathcal{G}}^{(\mathrm{NS})}_{\tilde{I},s,w_0}(\bar{\tau},\bar{z}) &= \frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} q^{\frac{\mathbf{n}}{2}a^2}e^{2\pi i\mathbf{n}\bar{z}a}\,\tilde{F}^{(\mathrm{NS})}_{\tilde{I}}(\bar{\tau},\bar{z}+a\bar{\tau}+b)\times \\ &\qquad\times {\chi_{\mathbf{dis}}}^{(\mathrm{NS})}_{(N,K)}(s,-Kn_I-Nw_0;\bar{\tau},-\bar{z}-a\bar{\tau}-b)\,, \end{aligned} \tag{3.29}$$

where we again assumed (3.19). The desired $\widetilde{\mathrm{R}}$-parts $\mathcal{G}^{(\widetilde{\mathrm{R}})}_{I,s,w_0}(\tau,z)$, $\tilde{\mathcal{G}}^{(\widetilde{\mathrm{R}})}_{\tilde{I},s,w_0}(\bar{\tau},\bar{z})$ are defined by the 1/2-spectral flow as in (3.18). Writing the Witten indices as

$$\lim_{\bar{z}\to 0} \tilde{\mathcal{G}}^{(\widetilde{\mathrm{R}})}_{\tilde{I},s,w_0}(\bar{\tau},\bar{z}) = \mathcal{I}_{\tilde{I},s,w_0}\,, \tag{3.30}$$

we obtain the general formula of elliptic genus

$$\mathcal{Z}(\tau,z) = \frac{1}{N}\sum_{I,\tilde{I}}\sum_{s=K}^{N+K}\sum_{w_0\in\mathbb{Z}_{2K}} a(s) N_{I,\tilde{I}}\mathcal{I}_{\tilde{I},s,w_0}\,\mathcal{G}^{(\widetilde{\mathrm{R}})}_{I,s,w_0}(\tau,z)\,. \tag{3.31}$$

For our later calculations it is useful to note the formula of Witten index (B.11), that is,

$$\lim_{z\to 0}\,\chi_{\mathbf{dis}}{}^{(\widetilde{\mathrm{R}})}_{(N,K)}(s,m;\tau,z) = -\delta^{(2NK)}_{m,s-K}\,. \tag{3.32}$$

It is also useful to introduce

$$\mathcal{Z}_{N,K}(\tau,z) \equiv -\sum_{s=K}^{N+K} a(s)\chi_{\mathbf{dis}}{}^{(\widetilde{\mathrm{R}})}_{(N,K)}(s,s-K;\tau,z)\,, \tag{3.33}$$

which describes the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$-part of elliptic genera in the prescription of $\mathbb{Z}_N$-orbifoldization. With the help of (B.10) we can further rewrite it as

$$\begin{aligned}
\mathcal{Z}_{N,K}(\tau,z) &= -\sum_{s=K}^{N+K}\sum_{n\in\mathbb{Z}} a(s)\,\frac{\left(e^{2\pi i z}q^{Nn}\right)^{\frac{s-K}{N}}}{1-e^{2\pi i z}q^{Nn}}\,e^{i\pi 4Knz}q^{NKn^2}\,\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\\
&= -\sum_{n\in\mathbb{Z}}\frac{e^{i\pi 4Knz}q^{NKn^2}}{1-e^{\frac{2\pi i}{N}z}q^n}\,\frac{i\theta_1(\tau,z)}{\eta(\tau)^3} + \frac{1}{2}\Theta_{0,NK}\left(\tau,\frac{2z}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\\
&\equiv -\left[\mathcal{K}_{2NK}\left(\tau,\frac{z}{N},0\right) - \frac{1}{2}\Theta_{0,NK}\left(\tau,\frac{2z}{N}\right)\right]\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,,
\end{aligned} \tag{3.34}$$

where $\mathcal{K}_\ell(\tau,\nu,\mu)$ is the "level ℓ Appell function" [35, 36] defined by

$$\mathcal{K}_\ell(\tau,\nu,\mu) \equiv \sum_{m\in\mathbb{Z}}\frac{e^{i\pi m^2\ell\tau+2\pi i m\ell\nu}}{1-e^{2\pi i(\nu+\mu+m\tau)}}\,. \tag{3.35}$$

We present examples of concrete calculations:

Example 1. Conifold ($N = K = 1$, $\mathcal{M}$ is trivial, $\mathbf{n} = 3$): this is the simplest example and should be identified with the deformed conifold (under the T-duality to $\mathcal{N}=2$ Liouville) [1]. The elliptic genus (3.31) has a simple form

$$\mathcal{Z}_{\mathrm{conifold}}(\tau,z) = \mathcal{Z}_{1,1}(\tau,z) = -\frac{1}{2}\left(\chi_{\mathbf{dis}}{}^{(\widetilde{\mathrm{R}})}_{(1,1)}(1,0;\tau,z) + \chi_{\mathbf{dis}}{}^{(\widetilde{\mathrm{R}})}_{(1,1)}(2,1;\tau,z)\right)\,. \tag{3.36}$$

We only have the boundary terms of the range (2.58). The following identity is presented in [54] and quite useful;

$$\begin{aligned}
\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q=\pm 1;\tau,z) &\equiv \sum_{n\in\mathbb{Z}}\frac{q^{n^2-n+1/4}e^{\pm 2\pi i(2n-1)z}}{1+e^{\pm 2\pi i z}q^{n-1/2}}\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\\
&= \pm\frac{1}{2}\left(\frac{\Theta_{1,3/2}(\tau,2z)}{\eta(\tau)} - \frac{\Theta_{-1,3/2}(\tau,2z)}{\eta(\tau)}\right) + \frac{1}{2}\Theta_{1,1}(\tau,2z)\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,,
\end{aligned} \tag{3.37}$$

where $\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q;\tau,z)$ is the characters of $\hat{c}=3$ extended chiral algebra [54, 49] (we use the notation given in appendix C of [18]). Applying the 1/2-spectral flow $z \rightarrow z+\frac{\tau}{2}+\frac{1}{2}$ to both sides of (3.37), we obtain the identity (see also [36])

$$\begin{aligned}
&\chi_{\mathbf{dis}}{}^{(\tilde{\mathrm{R}})}_{(1,1)}(1,0;\tau,z)\left(\equiv \chi_{\mathbf{dis}}{}^{(\tilde{\mathrm{R}})}_{(1,1)}(2,1;\tau,-z)\right) \equiv \\
&\quad \equiv \mathcal{K}_2(\tau,z,0)\,\frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \\
&\quad = -\frac{1}{2}\frac{1}{\eta(\tau)}\left(\tilde{\Theta}_{-1/2,3/2}(\tau,2z)+\tilde{\Theta}_{1/2,3/2}(\tau,2z)\right)+\frac{1}{2}\Theta_{0,1}(\tau,2z)\,\frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \\
&\quad = -\frac{1}{2}\frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)}+\frac{1}{2}\Theta_{0,1}(\tau,2z)\,\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,.
\end{aligned} \tag{3.38}$$

To derive the last line we used the Watson's quintuple product identity (A.7) (see e.g. [55].) In this way we obtain the simple formula

$$\boxed{\mathcal{Z}_{\mathrm{conifold}}(\tau,z)=\frac{1}{2}\frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)}\,.} \tag{3.39}$$

It may be worthwhile to point out the following fact: The elliptic genus for the level $n-2$ $\mathcal{N}=2$ minimal model was calculated in [51] by using the free field method in $\mathcal{N}=2$ Landau-Ginzburg theory with superpotential $W(X)=X^n$. The result is expressed as

$$\mathcal{Z}_{M_{n-2}}(\tau,z)=\sum_{\ell=0}^{n-2}\mathrm{ch}^{(\tilde{\mathrm{R}})}_{\ell,\ell+1}(\tau,z)=-\sum_{\ell=0}^{n-2}\mathrm{ch}^{(\tilde{\mathrm{R}})}_{\ell,-(\ell+1)}(\tau,z)=\frac{\theta_1(\tau,\frac{n-1}{n}z)}{\theta_1(\tau,\frac{1}{n}z)}\,, \tag{3.40}$$

where $\mathrm{ch}^{(\tilde{\mathrm{R}})}_{\ell,m}(\tau,z)$ is the character of minimal model M_{n-2} (A.6). It is curious to see that the "analytic continuation" of this formula to the inverse power potential $W(X)=X^{-1}$ $(n=-1)$, which is often used to describe the conformal system of conifold [1], correctly reproduces our result (3.39) (up to normalization).

Example 2. ALE(A_{n-1}) ($\mathcal{M}=M_{n-2}$, $N=n$, $K=1$, $\mathbf{n}=2$): this is the conformal system first analyzed in [2] and is considered to describe the ALE space obtained by deforming the A_{n-1}-type singularity (in the case of the diagonal modular invariants in M_{n-2}). The formula (3.31) gives us

$$\boxed{\mathcal{Z}_{\mathrm{ALE}(A_{n-1})}(\tau,z)=\sum_{\ell=0}^{n-2}\sum_{\substack{m\in\mathbb{Z}_{2n}\\ \ell+m\equiv 1 \pmod 2}}\mathrm{ch}^{(\tilde{\mathrm{R}})}_{\ell,m}(\tau,z)\,\chi_{\mathbf{dis}}{}^{(\tilde{\mathrm{R}})}_{(n,1)}(\ell+2,-m;\tau,z)} \tag{3.41}$$

Note that only the representations with $2\leq s(\equiv \ell+2)\leq n$ contributes in this case, and thus we do not have the boundary terms in contrast with the Example 1. The Witten index is evaluated as

$$\lim_{z\to 0}\mathcal{Z}_{\mathrm{ALE}(A_{n-1})}(\tau,z)=n-1\,, \tag{3.42}$$

JHEP05(2004)014

which reproduces the correct number of non-contractable 2-cycles. In the special case of Eguchi-Hanson space ALE(A_1) the formula (3.41) is reduced to

$$
\begin{aligned}
\mathcal{Z}_{\mathrm{ALE}(A_1)}(\tau,z) &= \chi_{\mathbf{dis}}{}^{(\widetilde{\mathrm{R}})}_{(2,1)}(2,-1;\tau,z) - \chi_{\mathbf{dis}}{}^{(\widetilde{\mathrm{R}})}_{(2,1)}(2,1;\tau,z) \\
&= -\mathrm{ch}_0^{\mathcal{N}=4\,(\widetilde{\mathrm{R}})}(\ell=0;\tau,z) \\
&\equiv -\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{1}{2}n(n+1)} e^{2\pi i\left(n+\frac{1}{2}\right)z}}{1-e^{2\pi i z}q^n} \, \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,,
\end{aligned} \tag{3.43}
$$

where $\mathrm{ch}_0^{\mathcal{N}=4\,(\widetilde{\mathrm{R}})}(\ell;\tau,z)$ denotes the level 1 $\mathcal{N}=4$ massless character of spin $\ell(=0,1/2)$ (related with $\mathrm{ch}_0^{\mathcal{N}=4\,(\mathrm{NS})}(1/2-\ell;\tau,z)$ by the spectral flow) [56].

Using the formulas (3.34) and (3.40), we can also rewrite (3.41) in the form that makes the orbifold structure manifest as in [53];

$$
\boxed{
\begin{aligned}
\mathcal{Z}_{\mathrm{ALE}(A_{n-1})}(\tau,z) &= \frac{1}{n}\sum_{a,b\in\mathbb{Z}_n} q^{a^2} e^{i4\pi a z}\, \mathcal{Z}_{M_{n-2}}(\tau,z)\mathcal{Z}_{n,1}(\tau,z) \\
&= -\frac{1}{n}\sum_{a,b\in\mathbb{Z}_n} (-1)^{a+b} q^{\frac{a^2}{2}} e^{i2\pi a z}\, \frac{\theta_1\left(\tau,\frac{n-1}{n}(z+a\tau+b)\right)}{\theta_1\left(\tau,\frac{1}{n}(z+a\tau+b)\right)} \times \\
&\qquad \times \mathcal{K}_{2n}\left(\tau,\frac{1}{n}(z+a\tau+b),0\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}
\end{aligned}
} \tag{3.44}
$$

Here we used the relation $\theta_1(\tau,z+a\tau+b) = (-1)^{a+b} q^{-\frac{a^2}{2}} e^{-i2\pi a z}\,\theta_1(\tau,z)$. Note that the theta function term in (3.34) is eliminated by the $\mathbb{Z}_N$-orbifoldization.

One can also perform similar calculations in the cases of CY_4-singularity of A_{n-1}-type ($N=n$, $K=n+1$). We obtain

$$
\boxed{
\begin{aligned}
\mathcal{Z}_{CY_4\,(A_{n-1})}(\tau,z) &= \sum_{\ell=0}^{n-2}\sum_{\substack{m\in\mathbb{Z}_{2n}\\ \ell+m\equiv 1 \pmod 2}} \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell,m}(\tau,z)\, \chi_{\mathbf{dis}}{}^{(\widetilde{\mathrm{R}})}_{(n,n+1)}(\ell+n+2,-m;\tau,z) \\
&= -\frac{1}{n}\sum_{a,b\in\mathbb{Z}_n} (-1)^{a+b} q^{\frac{3a^2}{2}} e^{i6\pi a z}\, \frac{\theta_1\left(\tau,\frac{n-1}{n}(z+a\tau+b)\right)}{\theta_1\left(\tau,\frac{1}{n}(z+a\tau+b)\right)} \\
&\qquad \times \mathcal{K}_{2n(n+1)}\left(\tau,\frac{1}{n}(z+a\tau+b),0\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}
\end{aligned}
} \tag{3.45}
$$

Example 3. General cases of non-compact CY_3: lastly we consider the general models of $\mathbf{n}=3$, where $\mathcal{M}$ is an arbitrary $\mathcal{N}=2$ RCFT with $\hat{c}_{\mathcal{M}}<2$. We make a natural assumption of the "charge conjugation symmetry" in the $\mathcal{M}$-sector. Namely, we postulate

$$
N_{c(I),c(\tilde{I})} = N_{I,\tilde{I}}\,, \tag{3.46}
$$

where the charge conjugation $c: I\to c(I)$ is defined by

$$
F^{(\mathrm{NS})}_{c(I)}(\tau,z) \equiv F^{(\mathrm{NS})}_{I}(\tau,-z)\,. \tag{3.47}
$$

Rather surprisingly we can show that the elliptic genera for these models have a simple and universal form;

$$\boxed{\mathcal{Z}(\tau,z)=\frac{\chi}{2}\left(\frac{\tilde{\Theta}_{-1/2,3/2}(\tau,2z)}{\eta(\tau)}+\frac{\tilde{\Theta}_{1/2,3/2}(\tau,2z)}{\eta(\tau)}\right)\equiv\frac{\chi}{2}\frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)}} \tag{3.48}$$

where $\chi \equiv \lim_{z\to 0}\mathcal{Z}(\tau,z)$ is the Witten index that counts the Ramond ground states in the total system.

We sketch how one can derive this formula. Thanks to the above assumption (3.46), the elliptic genus is found to have only contributions of the symmetrized forms $\mathcal{G}^{(\widetilde{\mathrm{R}})}_{I,s,w_0}(\tau,z)+\mathcal{G}^{(\widetilde{\mathrm{R}})}_{I,s,w_0}(\tau,-z)$. Furthermore, because of the U(1)-projection the NS conformal blocks $\mathcal{G}^{(\mathrm{NS})}_{I,s,w_0}$ should be expanded with positive integer coefficients by the characters of the $\hat{c}=3$ extended algebra [54]: $\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q=1;\tau,z)$, $\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q=-1;\tau,z)$, and massive characters

$$\begin{aligned}\mathrm{Ch}^{(\mathrm{NS})}(h,Q=0;\tau,z) &\equiv q^{h-1/4}\Theta_{0,1}(\tau,2z)\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,,\\ \mathrm{Ch}^{(\mathrm{NS})}(h,|Q|=1;\tau,z) &\equiv q^{h-1/2}\Theta_{1,1}(\tau,2z)\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,.\end{aligned} \tag{3.49}$$

(We again use the notations given in appendix C of [18].) Note that the graviton representation ($h=Q=0$) cannot occur owing to the constraint $K\leq s\leq N+K$ (or (2.58)). Consequently, recalling the identity (3.37), (3.38), one can find

$$\mathcal{G}^{(\widetilde{\mathrm{R}})}_{I,s,w_0}(\tau,z)=(\text{integer})\times\frac{1}{2}\left(\frac{\tilde{\Theta}_{-1/2,3/2}(\tau,2z)}{\eta(\tau)}+\frac{\tilde{\Theta}_{1/2,3/2}(\tau,2z)}{\eta(\tau)}\right)+(\text{massive part})\,. \tag{3.50}$$

The massive part is generically an infinite sum of the terms of the forms $q^*\Theta_{0\,(1),1}(\tau,2z)\times i\theta_1(\tau,z)/\eta(\tau)^3$ and hence odd functions of z. It does not contribute because of the above remark. In this way we arrive at the general formula (3.48).

It has been shown in [53] that the elliptic genera for arbitrary Gepner models (or the LG-orbifolds) of compact CY_3 can be written in the form

$$\mathcal{Z}_{\mathrm{Gepner}}(\tau,z)=(h_{1,2}-h_{1,1})\left(\frac{\tilde{\Theta}_{-1/2,3/2}(\tau,2z)}{\eta(\tau)}+\frac{\tilde{\Theta}_{1/2,3/2}(\tau,2z)}{\eta(\tau)}\right), \tag{3.51}$$

where $h_{1,2}$, $h_{1,1}$ are the numbers of (c,c), (c,a)-type chiral primaries of $h=\tilde{h}=1/2$ respectively, which should be identified with the Hodge numbers of CY_3. Our result (3.48) is the generalization of this formula to the non-compact models. Note that $\chi=2(h_{1,2}-h_{1,1})$ is an even number for any compact CY_3, while χ in (3.48) is allowed to be odd. Recall the conifold case, Example 1.

A few remarks are in order:

1. The elliptic genus includes contributions of Ramond ground states that are naively supposed to describe massless excitations in string theory. However, to identify them

JHEP05(2004)014

with the massless spectrum we must take account of the GSO condition for spin structures. The simplest way to do so is to look for the corresponding NS (anti-)chiral states with $h = \frac{1}{2}|Q| = \frac{1}{2}$. For example, in the ALE(A_{n-1}) case (Example 2), we have $n-1$ Ramond ground states and all of them correspond to massless string states. They are identified as each of normalizable deformations of A_{n-1}-singularity ("moduli"). However, even though the A_{n-1}-model of the Calabi-Yau 4-fold $\mathbf{n} = 4$ ($N = n$, $K = n+1$) still has the Witten index $\mathcal{Z}(\tau, 0) = n-1$, they cannot define NS massless states at all. This feature has its origin in the restriction $K \leq s \leq N + K$. We thus find the inequality

$$h \equiv \frac{s}{2N} \geq \frac{K}{2N} \equiv \frac{1}{2} + \frac{1}{2n} > \frac{1}{2}, \tag{3.52}$$

for the chiral primary states in the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-sector, implying no massless states appear in the superstring spectrum. These missing massless states would be associated to the *non-normalizable* deformations of A_{n-1}-singularity ("coupling constants"). In the Calabi-Yau 3-fold $\mathbf{n} = 3$ case the aspect of massless states is more complex: the half of them appears as massless states in the closed string spectrum and the remaining ones are missing. These aspects of marginal fields in singular $CY_{\mathbf{n}}$ have been discussed in [57, 58, 4, 6].

2. As we already mentioned, the elliptic genera of $\mathcal{N} = 2$ RCFT's for compact $CY_{\mathbf{n}}$ are known to be the (weak) Jacobi form with weight 0 and index $\mathbf{n}/2$. This means that the elliptic genera have the following properties;

$$q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a z}\, \mathcal{Z}(\tau, z + a\tau + b) = (-1)^{\mathbf{n}(a+b)}\, \mathcal{Z}(\tau, z)\,, \qquad ({}^\forall a, b \in \mathbb{Z})\,, \tag{3.53}$$

$$\mathcal{Z}(\tau + 1, z) = \mathcal{Z}(\tau, z)\,, \tag{3.54}$$

$$\mathcal{Z}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi \mathbf{n} \frac{z^2}{\tau}}\, \mathcal{Z}(\tau, z)\,. \tag{3.55}$$

It is easy to confirm that our elliptic genera for non-compact models (3.31) satisfy (3.53) and (3.54). However, the third condition for the S-transformation is not necessarily obeyed. For the $\mathbf{n} = 3$ cases, (3.55) is also satisfied because of the general formula (3.48). However, in the $\mathbf{n} = 2, 4$ cases, the elliptic genera are proportional to the Appell function which transforms as [36]

$$\begin{aligned}\mathcal{K}_\ell\left(-\frac{1}{\tau}, \frac{\nu}{\tau}, \frac{\mu}{\tau}\right) &= \tau e^{i\pi\ell\frac{\nu^2-\mu^2}{\tau}}\, \mathcal{K}_\ell(\tau, \nu, \mu) + \\ &\quad + \tau \sum_{a=0}^{\ell-1} e^{i\pi\frac{\ell}{\tau}(\nu + \frac{a}{\ell}\tau)^2}\, \Phi(\ell\tau, \ell\mu - a\tau)\theta_3(\ell\tau, \ell\nu + a\tau)\end{aligned} \tag{3.56}$$

where

$$\Phi(\tau, \mu) = -\frac{i}{2\sqrt{-i\tau}} - \frac{1}{2}\int_{-\infty}^{\infty} dx\, e^{-\pi x^2} \frac{\sinh(\pi x\sqrt{-i\tau}(1 + 2\frac{\mu}{\tau}))}{\sinh(\pi x\sqrt{-i\tau})} \tag{3.57}$$

The above transformation law corresponds to the mixing of discrete and continuous representations in $\mathcal{N} = 2$ Liouville theory [56, 54, 59, 18] (see appendix B). Appell

JHEP05(2004)014

functions (3.35) are interpreted as sections of higher rank vector bundles over elliptic curves as opposed to theta-functions which are sections of line bundles [35]. It is interesting to see if one can achieve a precise geometrical interpretation of the elliptic genera in the non-compact space-time.

4. Summary

In this paper we have confirmed the correspondence between $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1)$ supercoset and $\mathcal{N}=2$ Liouville theory and computed the elliptic genera for various singular space-times. We summarize the main ingredients of this paper.

1. Partition functions of $\mathrm{SL}(2;\mathbb{R})_k/\mathrm{U}(1)$ theories are decomposed into two pieces: (1) the part consisting of continuous (massive) representations and (2) the part consisting of discrete (massless matter) representations.

2. The continuous part is proportional to the volume factor, since it describes the propagating modes in the bulk, and gives the leading contribution to the partition function. It seems that strictly modular invariant partition functions are obtained after the division by the infinite volume factor. Then they contain only the continuous representations (massive modes), and reproduce the results obtained previously in [7].

3. The discrete part describes excitations localized around the tip of cigar [33] and thus appears without the volume factor. Embedded in superstring vacua it could be interpreted as contributions from massless matter fields corresponding to the deformations of Calabi-Yau singularities.

4. Continuous representations do not contribute to the elliptic genera and thus elliptic genus clearly exhibits the contributions of the discrete states. In generic cases (of $\hat{c}_{\mathrm{tot}}=2,4$) the elliptic genera possess complex modular behaviors, and they are not Jacobi forms (section of line bundles) but sections of higher rank vector bundles. On the other hand, in models with $\hat{c}_{\mathrm{tot}}=3$ the elliptic genera behave in the same way as rational conformal theories.

5. When embedded in superstring vacua by means of the Gepner-like method, the extended characters defined in [18] emerges quite naturally in continuous and discrete series (see (3.12), (3.26)). We have also confirmed that discrete representations in $\mathcal{N}=2$ Liouville theory closed under modular transformations are mapped to unitary discrete representations in the range $1/2 \leq j \leq (k+1)/2$ which appear in the regularized partition function of $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ theory. This justifies our Ansatz for the basis of Ishibashi states of $\mathcal{N}=2$ Liouville theory we have proposed in [18].

6. It appears quite likely that $\mathcal{N}=2$ Liouville and $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ theories are in fact exactly mapped into each other (T-duality) and will essentially be one and the same theory with identical physical contents. This is gratifying since in the Liouville approach it has been extremely difficult to incorporate the effects of the Liouville potential terms non-perturbatively into the theory. We note that in the $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$

JHEP05(2004)014

supercoset theory, on the other hand, the space-time is curved into a 2D black hole but the (cosmological constant) parameter μ does not appear explicitly. Thus it seems that $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ theory has deformed the space-time by absorbing the Liouville potential terms. Agreement of our Liouville results [18] with those of $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ theory is encouraging and indicates that we have incorporated properly the effects of Liouville potential terms in the analysis.

Acknowledgments

We are grateful to A. Taormina for her correspondence about the higher level Appell functions. We also would like to thank Y. Nakayama for valuable discussions.

The research of T.E. and Y.S. is partially supported by Japanese Ministry of Education, Culture, Sports, Science and Technology.

A. Notations and some useful formulas

A.1 Theta functions

We here summarize our notations of theta functions. We set $q \equiv e^{2\pi i\tau}$ and $y \equiv e^{2\pi iz}$,

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \\
&\equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m), \\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \\
&\equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m), \\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^{n} \\
&\equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}), \\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^{n} \\
&\equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}),
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\Theta_{m,k}(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}, \\
\tilde{\Theta}_{m,k}(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}. \\
\eta(\tau) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n).
\end{aligned} \tag{A.2}$$

JHEP05(2004)014

A.2 Character formulas for the $\mathcal{N}=2$ minimal models

The easiest way to represent the character formulas of the level k $\mathcal{N}=2$ minimal model ($\hat{c}=k/(k+2)$) is to use its realization as the coset $\frac{\mathrm{SU}(2)_k\times \mathrm{SO}(2)_1}{\mathrm{U}(1)_{k+2}}$. We then have the following branching relation;

$$\begin{aligned}\chi_\ell^{(k)}(\tau,w)\Theta_{s,2}(\tau,w-z) &= \sum_{\substack{m\in\mathbb{Z}_{2(k+2)}\\ \ell+m+s\in 2\mathbb{Z}}} \chi_m^{\ell,s}(\tau,z)\Theta_{m,k+2}(\tau,w-2z/(k+2)),\\ \chi_m^{\ell,s}(\tau,z) &\equiv 0, \qquad \text{for } \ell+m+s\in 2\mathbb{Z}+1,\end{aligned} \tag{A.3}$$

where $\chi_\ell^{(k)}(\tau,z)$ is the spin $\ell/2$ character of $\mathrm{SU}(2)_k$;

$$\chi_\ell^{(k)}(\tau,z)=\frac{\Theta_{\ell+1,k+2}(\tau,z)-\Theta_{-\ell-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z)-\Theta_{-1,2}(\tau,z)} \equiv \sum_{m\in\mathbb{Z}_{2k}} c_{\ell,m}^{(k)}(\tau)\Theta_{m,k}(\tau,z)\,. \tag{A.4}$$

The branching function $\chi_m^{\ell,s}(\tau,z)$ is explicitly calculated as follows;

$$\chi_m^{\ell,s}(\tau,z)=\sum_{r\in\mathbb{Z}_k} c_{\ell,m-s+4r}^{(k)}(\tau)\Theta_{2m+(k+2)(-s+4r),2k(k+2)}(\tau,z/(k+2))\,. \tag{A.5}$$

Then, the desired character formulas are written as

$$\begin{aligned}\mathrm{ch}_{\ell,m}^{(\mathrm{NS})}(\tau,z) &= \chi_m^{\ell,0}(\tau,z)+\chi_m^{\ell,2}(\tau,z)\,,\\ \mathrm{ch}_{\ell,m}^{(\widetilde{\mathrm{NS}})}(\tau,z) &= \chi_m^{\ell,0}(\tau,z)-\chi_m^{\ell,2}(\tau,z)\equiv e^{-i\pi\frac{m}{k+2}}\mathrm{ch}_{\ell,m}^{(\mathrm{NS})}\left(\tau,z+\frac{1}{2}\right),\\ \mathrm{ch}_{\ell,m}^{(\mathrm{R})}(\tau,z) &= \chi_m^{\ell,1}(\tau,z)+\chi_m^{\ell,3}(\tau,z)\equiv q^{\frac{k}{8(k+2)}}y^{\frac{k}{2(k+2)}}\mathrm{ch}_{\ell,m+1}^{(\mathrm{NS})}\left(\tau,z+\frac{\tau}{2}\right),\\ \mathrm{ch}_{\ell,m}^{(\widetilde{\mathrm{R}})}(\tau,z) &= \chi_m^{\ell,1}(\tau,z)-\chi_m^{\ell,3}(\tau,z)\equiv -e^{-i\pi\frac{m+1}{k+2}}q^{\frac{k}{8(k+2)}}y^{\frac{k}{2(k+2)}}\mathrm{ch}_{\ell,m+1}^{(\mathrm{NS})}\left(\tau,z+\frac{1}{2}+\frac{\tau}{2}\right).\end{aligned} \tag{A.6}$$

By definition, we restrict to $\ell+m\in 2\mathbb{Z}$ in NS and $\widetilde{\mathrm{NS}}$ sectors, and to $\ell+m\in 2\mathbb{Z}+1$ in R and $\widetilde{\mathrm{R}}$ sectors.

A.3 Useful identity

$$\begin{aligned}&\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1})(1-y^2q^{2n-1})(1-y^{-2}q^{2n-1})=\\ &\qquad =\sum_{m\in\mathbb{Z}}\left(y^{3m}-y^{-3m-1}\right)q^{\frac{1}{2}m(3m+1)},\\ \Longleftrightarrow\ &\frac{1}{\eta(\tau)}\left(\tilde{\Theta}_{1/2,3/2}(\tau,2z)+\tilde{\Theta}_{-1/2,3/2}(\tau,2z)\right)=\frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)}\end{aligned} \tag{A.7}$$

(Watson's quintuple product identity).

The following identity is often useful in checking the modular invariance

$$\frac{\left(\mathrm{Im}\left(\frac{u}{\tau}\right)\right)^2}{\mathrm{Im}\left(-\frac{1}{\tau}\right)}=\frac{(\mathrm{Im}\,u)^2}{\mathrm{Im}\,\tau}+i\frac{u^2}{2\tau}-i\frac{\bar{u}^2}{2\bar{\tau}}\,. \tag{A.8}$$

We also note that the combination $|u|^2/\tau_2$ is modular invariant.

JHEP05(2004)014

B. $\mathcal{N}=2$ (extended) character formulas for $\hat{c}>1$

We denote the conformal weight and U(1)-charge of the highest weight state as h, Q and again set $q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi i z}$. The irreducible characters of $\mathcal{N}=2$ SCA with $\hat{c}>1$ are summarized as follows [60];

1. *Massive representations:*

$$\mathrm{ch}^{(\mathrm{NS})}(h,Q;\tau,z) = q^{h-(\hat{c}-1)/8}\, y^Q\, \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \qquad (h>|Q|/2,\ 0\le |Q| < \hat{c}-1)\,. \tag{B.1}$$

2. *Massless matter representations:*

$$\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q;\tau,z) = q^{\frac{|Q|}{2}-(\hat{c}-1)/8} y^Q\, \frac{1}{1+y^{\mathrm{sgn}(Q)}q^{1/2}}\, \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,. \tag{B.2}$$

They correspond to the (anti-)chiral primary state with $h=|Q|/2$, $(0<|Q|<\hat{c})$.

3. *Graviton representation:*

$$\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(\tau,z) = q^{-(\hat{c}-1)/8}\, \frac{1-q}{(1+yq^{1/2})(1+y^{-1}q^{1/2})}\, \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,. \tag{B.3}$$

They correspond to the vacuum $h=Q=0$, which is the unique state being both chiral and anti-chiral primary.

More general unitary representations are generated by the integral spectral flows and classified in [41]. The spectral flow generator U_η with a real parameter η is defined by

$$\begin{aligned} U_\eta^{-1} L_m U_\eta &= L_m + \eta J_m + \frac{\hat{c}}{2}\eta^2 \delta_{m,0}\,,\\ U_\eta^{-1} J_m U_\eta &= J_m + \hat{c}\eta\delta_{m,0}\,,\\ U_\eta^{-1} G_r^{\pm} U_\eta &= G^{\pm}_{r\pm\eta}\,. \end{aligned} \tag{B.4}$$

Half-integral spectral flows $\eta \in \frac{1}{2}+\mathbb{Z}$ intertwine the NS and R sector characters, while the integral spectral flows $\eta = n \in \mathbb{Z}$ keep the spin structure. The spectrally flowed characters are given by

$$\mathrm{ch}^{(\mathrm{NS})}_{*}(*,n;\tau,z) \equiv q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n}\, \mathrm{ch}^{(\mathrm{NS})}_{*}(*;\tau,z+n\tau)\,, \qquad (n\in\mathbb{Z})\,, \tag{B.5}$$

where $\mathrm{ch}^{(\mathrm{NS})}_{*}(*;\tau,z)$ is the abbreviated notation of (B.1)–(B.3).

For the theory of $\hat{c} = 1+2K/N$ $(N,K\in\mathbb{Z}_{>0})$, we introduce the "extended characters" [18] which should be the characters of unitary representations of the extended chiral algebra defined by adding the spectral flow generators $U_{\pm N}$ [56, 54, 49, 50].

$$\begin{aligned} \mathrm{Ch}^{(\mathrm{NS})}(h,\alpha;\tau,z) &= \sum_{n\in r+N\mathbb{Z}} q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n}\, \mathrm{ch}^{(\mathrm{NS})}\left(h_0, Q=\frac{\alpha_0}{N};\tau,z+n\tau\right)\\ &\equiv q^{p^2/2}\Theta_{\alpha,NK}\left(\tau,\frac{2z}{N}\right)\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \end{aligned} \tag{B.6}$$

$$\left(h \equiv h_0 + \frac{r\alpha_0}{N} + \frac{Kr^2}{N} \equiv \frac{p^2}{2} + \frac{\alpha^2+K^2}{4NK}\,,\ \alpha \equiv \alpha_0 + 2Kr\right),$$

$$\begin{aligned}
\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(r,s;\tau,z) &= \sum_{n\in r+N\mathbb{Z}} q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n}\, \mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}\left(Q=\frac{s}{N};\tau,z+n\tau\right),\\
&\equiv \sum_{m\in\mathbb{Z}} \frac{\left(yq^{N\left(m+\frac{2r+1}{2N}\right)}\right)^{\frac{s-K}{N}}}{1+yq^{N\left(m+\frac{2r+1}{2N}\right)}} y^{2K\left(m+\frac{2r+1}{2N}\right)} q^{NK\left(m+\frac{2r+1}{2N}\right)^2} \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \qquad (B.7)\\
\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{G}}(r;\tau,z) &= \sum_{n\in r+N\mathbb{Z}} q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n}\, \mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(\tau,z+n\tau)\,,\\
&\equiv q^{-\frac{K}{4N}} \sum_{m\in\mathbb{Z}} q^{NK\left(m+\frac{r}{N}\right)^2+N\left(m+\frac{2r-1}{2N}\right)} y^{2K\left(m+\frac{r}{N}\right)+1} \times\\
&\quad\times \frac{1-q}{\left(1+yq^{N\left(m+\frac{2r+1}{2N}\right)}\right)\left(1+yq^{N\left(m+\frac{2r-1}{2N}\right)}\right)} \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \qquad (B.8)
\end{aligned}$$

where the ranges of parameters r, α, s are given as

$$r\in\mathbb{Z}_N\,, \qquad \alpha\in\mathbb{Z}_{2NK}\,, \qquad 1\le s\le N+2K-1\,, \qquad (s\in\mathbb{Z})\,. \qquad (B.9)$$

In the calculations of elliptic genera, we use the following formula ($r\in\frac{1}{2}+\mathbb{Z}_N$)

$$\begin{aligned}
\mathrm{Ch}^{(\widetilde{\mathrm{R}})}_{\mathbf{M}}(r,s;\tau,z) &\equiv e^{-i\pi\frac{s+K(2r-1)}{N}} q^{\frac{\hat{c}}{8}} y^{\frac{\hat{c}}{2}}\, \mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}\left(r-\frac{1}{2},s;\tau,z+\frac{\tau}{2}+\frac{1}{2}\right)\\
&= \sum_{m\in\mathbb{Z}} \frac{\left(yq^{N\left(m+\frac{2r+1}{2N}\right)}\right)^{\frac{s-K}{N}}}{1-yq^{N\left(m+\frac{2r+1}{2N}\right)}} y^{2K\left(m+\frac{2r+1}{2N}\right)} q^{NK\left(m+\frac{2r+1}{2N}\right)^2} \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,, \qquad (B.10)
\end{aligned}$$

which yields the Witten index

$$\lim_{z\to 0} \mathrm{Ch}^{(\widetilde{\mathrm{R}})}_{\mathbf{M}}(r,s;\tau,z) = -\delta^{(N)}_{r,-1/2}\,. \qquad (B.11)$$

For convenience of readers we also present the modular transformation formulas given in [18] in terms of the new notations (3.12), (3.26). They are written as

$$\begin{aligned}
\chi_{\mathbf{con}}{}^{(\mathrm{NS})}\left(p,m;-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{i\pi\hat{c}\frac{z^2}{\tau}} \frac{2}{N} \sum_{m'\in\mathbb{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}} \times\\
&\quad\times \int_0^\infty dp'\, \cos\left(2\pi\frac{2K}{N}pp'\right) \chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p',m';\tau,z)\,, \qquad (B.12)\\
\chi_{\mathbf{dis}}{}^{(\mathrm{NS})}\left(s,m;-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{i\pi\hat{c}\frac{z^2}{\tau}} \left[\frac{1}{N}\sum_{m'\in\mathbb{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}} \times\right.\\
&\quad\times \int_0^\infty dp' \frac{\cosh(2\pi\frac{N-(s-K)}{N}p') + e^{2\pi i\frac{m'}{2K}}\cosh(2\pi\frac{s-K}{N}p')}{2\left|\cosh\pi\left(p'+i\frac{m'}{2K}\right)\right|^2} \times\\
&\quad\times \chi_{\mathbf{con}}{}^{(\mathrm{NS})}(p',m';\tau,z) +\\
&\quad+ \frac{i}{N}\sum_{s'=K+1}^{N+K-1}\sum_{m'\in\mathbb{Z}_{2NK}} e^{2\pi i\frac{(s-K)(s'-K)-mm'}{2NK}} \chi_{\mathbf{dis}}{}^{(\mathrm{NS})}(s',m';\tau,z) +
\end{aligned}$$

$$+\frac{i}{2N}\sum_{m'\in\mathbb{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}}\Big\{{\chi_{\mathbf{dis}}}^{(\mathrm{NS})}(K,m';\tau,z)- \tag{B.13}$$

$$-{\chi_{\mathbf{dis}}}^{(\mathrm{NS})}(N+K,m'+N;\tau,z)\Big\}\Bigg].$$

C. Explicit calculation of partition function

In this appendix we present the explicit derivation of (2.38) by the path-integration. This is almost parallel to the analysis of the bosonic models given in [30, 31]. We define the world-sheet torus by the identifications $(w,\bar{w})\sim(w+2\pi,\bar{w}+2\pi)\sim(w+2\pi\tau,\bar{w}+2\pi\bar{\tau})$ $(\tau\equiv\tau_1+i\tau_2,\ \tau_2>0$, and use the convention $z=e^{iw}$, $\bar{z}=e^{-i\bar{w}})$. We call the cycles defined by these two identifications as the α and β-cycles as usual.

The desired partition function is written as

$$Z(\tau)=\int\mathcal{D}[g,A,\psi^{\pm},\tilde{\psi}^{\pm}]\,e^{-\kappa S_{\mathrm{gWZW}}(g,A)-S_{\psi}(\psi^{\pm},\tilde{\psi}^{\pm},A)}\,. \tag{C.1}$$

Although the euclidean 2D BH is manifestly positive definite (since the time-like U(1) is gauged away), the calculation of partition function could be subtle due to the lorentzian signature in the parent $\mathrm{SL}(2;\mathbb{R})$ theory. Therefore, it is better to start with the Wick rotated model $H_3^+/\mathbb{R}$, where $H_3^+\cong\mathrm{SL}(2;\mathbb{C})/\mathrm{SU}(2)$ is the euclidean AdS_3. The Wick rotation is defined by the replacement; $g\in\mathrm{SL}(2;\mathbb{R})\rightarrow g\in\mathrm{SL}(2;\mathbb{C})/\mathrm{SU}(2)$ and the gauge field $A\equiv(A_{\bar{z}}d\bar{z}+A_z dz)\,\frac{\sigma_2}{2}$

To calculate the partition function (C.1) it is convenient to reexpress the gauged WZW action (2.4) in the form

$$S_{\mathrm{gWZW}}(g,A)=S_{\mathrm{WZW}}(h_L g h_R)-S_{\mathrm{WZW}}(h_L h_R^{-1})\equiv S^{(A)}_{\mathrm{gWZW}}(g,h_L,h_R)\,, \tag{C.2}$$

$$A_{\bar{z}}\frac{\sigma_2}{2}=\partial_{\bar{z}}h_L h_L^{-1}\,,\qquad A_z\frac{\sigma_2}{2}=\partial_z h_R h_R^{-1}\,, \tag{C.3}$$

where we used the abbreviated notation $S_{\mathrm{WZW}}(g)\equiv S^{\mathrm{SL}(2;\mathbb{R})}_{\mathrm{WZW}}(g)$. After the Wick rotation we must suppose $h_L=h_R^{\dagger}(\equiv h)\in\exp(\mathbb{C}\sigma_2)$. We also introduce the vector-like gauged WZW action

$$S^{(V)}_{\mathrm{gWZW}}(g,h_L,h_R)\equiv S_{\mathrm{WZW}}(h_L g h_R)-S_{\mathrm{WZW}}(h_L h_R)\,. \tag{C.4}$$

We can parameterize $h(\equiv h_L\equiv h_R^{\dagger})$ as

$$h=e^{(X+iY)\frac{\sigma_2}{2}}h^u\,,\qquad h^u\equiv e^{i\Phi^u\frac{\sigma_2}{2}}\,, \tag{C.5}$$

where $\Phi^u(w,\bar{w})$ is associated with the modulus of holomorphic line bundle; $u\equiv s_1\tau-s_2\in$ $\mathrm{Jac}(\Sigma)\cong\Sigma$, $(0\leq s_1,s_2<1)$, conventionally defined as

$$\Phi^u(w,\bar{w})=\frac{i}{2\tau_2}\left\{(w\bar{\tau}-\bar{w}\tau)s_1+(\bar{w}-w)s_2\right\}\,, \tag{C.6}$$

It is a real harmonic function satisfying the twisted boundary conditions

$$\begin{aligned}\Phi^u(w+2\pi,\bar{w}+2\pi) &= \Phi^u(w,\bar{w})+2\pi s_1\,,\\ \Phi^u(w+2\pi\tau,\bar{w}+2\pi\bar{\tau}) &= \Phi^u(w,\bar{w})+2\pi s_2\,.\end{aligned} \tag{C.7}$$

Real scalar fields X, Y correspond to the axial ($\mathbb{R}_A$) and vector ($\mathrm{U}(1)_V$) gauge transformations respectively. Using the Polyakov-Wiegmann identity

$$S^{(A)}_{\mathrm{gWZW}}(\Omega g\Omega^\dagger,\Omega^{-1}h,\Omega^{\dagger\,-1}h^\dagger)=S^{(V)}_{\mathrm{gWZW}}(g,h,h^\dagger)-S^{(A)}_{\mathrm{gWZW}}(\Omega^{-1}\Omega^\dagger,h,h^{\dagger\,-1})\,, \tag{C.8}$$

and the gauge invariance of path-integral measure $\mathcal{D}(\Omega g\Omega^\dagger)=\mathcal{D}g$, we can rewrite the partition function (C.1) as follows (after dividing by the gauge volume $\int\mathcal{D}X$);

$$\begin{aligned}Z(\tau) &= \int_\Sigma d^2u\int\mathcal{D}[g,Y,\psi^\pm,\tilde{\psi}^\pm,b,\tilde{b},c,\tilde{c}]\,e^{-\kappa S^{(V)}(g,h^u,h^{u\,\dagger})+(\kappa-2)S^{(A)}(e^{iY\sigma_2},h^u,h^{u\,\dagger\,-1})}\times\\ &\qquad\times e^{-S_\psi(\psi^\pm,\tilde{\psi}^\pm,a^u)-S_{\mathrm{gh}}(b,\tilde{b},c,\tilde{c})}\\ &\equiv \int_\Sigma d^2u\,Z_g(\tau,u)Z_Y(\tau,u)Z_\psi(\tau,u)Z_{\mathrm{gh}}(\tau)\,,\\ &\left(a^u_{\bar{w}}\equiv i\partial_{\bar{w}}\Phi^u(w,\bar{w})\equiv\frac{u}{2\tau_2}\,,\;a^u_w\equiv-i\partial_w\bar{\Phi}^u(w,\bar{w})\equiv\frac{\bar{u}}{2\tau_2}\right).\end{aligned} \tag{C.9}$$

Here b, c ($\tilde{b}$, $\tilde{c}$) are the spin (1,0) ghost system to rewrite the jacobian of path integral measure. The level shift $\kappa\to\kappa-2(\equiv k)$ for the action $S^{(A)}(e^{iY\sigma_2},h^u,h^{u\,\dagger\,-1})$ in (C.9) is owing to the chiral anomaly of the fermion determinant, regularized so that it is anomaly free along the axial direction.

We next evaluate each sector separately:

- *H_3^+-sector.*

  This non-trivial sector has been already evaluated in [30, 31] (see the comment below);

$$Z_g(\tau,u)\equiv\int\mathcal{D}g\,e^{-\kappa S^{(V)}(g,h^u,h^{u\,\dagger})}\propto\frac{e^{2\pi\frac{(\mathrm{Im}\,u)^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}\,. \tag{C.10}$$

  $Z_g(\tau,u)$ is indeed modular invariant, especially under the S-transformation $\tau\to-1/\tau$, $u\to u/\tau$. Note that the "anomaly factor" $e^{2\pi\frac{(\mathrm{Im}\,u)^2}{\tau_2}}$ remedies the S-invariance thanks to the identity (A.8) as mentioned in appendix B of [42].

- *$\mathrm{U}(1)_V$-sector.*

  Y is the coordinate along $\mathrm{U}(1)_V$-direction ($i\mathbb{R}\sigma_2$) and thus compact; $Y\sim Y+2\pi$. The relevant world-sheet action is calculated as

$$\begin{aligned}S_Y(Y;u) &\equiv -kS_{\mathrm{WZW}}(h^ue^{iY\sigma_2}h^{u\,\dagger\,-1})+kS_{\mathrm{WZW}}(h^uh^{u\,\dagger})\\ &= \frac{k}{\pi}\int d^2w\,|\partial_{\bar{w}}Y-ia^u_{\bar{w}}|^2=\frac{k}{\pi}\int d^2w\,|\partial_{\bar{w}}Y^u|^2\,.\end{aligned} \tag{C.11}$$

In the last line we set $Y^u \equiv Y + \Phi^u$, which satisfies the twisted boundary conditions;

$$\begin{aligned} Y^u(w+2\pi, \bar{w}+2\pi) &= Y^u(w,\bar{w}) + 2\pi(m+s_1)\,, \qquad (m \in \mathbb{Z}) \\ Y^u(w+2\pi\tau, \bar{w}+2\pi\bar{\tau}) &= Y^u(w,\bar{w}) + 2\pi(n+s_2)\,, \qquad (n \in \mathbb{Z})\,. \end{aligned} \tag{C.12}$$

Rescaling the twisted boson Y^u as $Y^u \to Y^u/\sqrt{2k}$, we arrive at the theory of a twisted compact boson with radius $R = \sqrt{2k}$. Therefore, the relevant partition function becomes

$$\begin{aligned} Z_Y(\tau,u) &= \int \mathcal{D}Y^u \, e^{-\frac{1}{2\pi}\int d^2w\, |\partial_w Y^u|^2} \\ &\propto \frac{1}{\sqrt{\tau_2}\,|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} \exp\left(-\frac{\pi k}{\tau_2}\,|(m+s_1)\tau - (n+s_2)|^2\right). \end{aligned} \tag{C.13}$$

Again this is manifestly modular invariant.

- *Fermion and ghost sectors.*

  The remaining fermionic sectors are easy to evaluate. They are the standard fermion determinants with anti-periodic and periodic boundary conditions respectively (for the NS sector of $\psi^\pm$)

$$Z_\psi^{(\mathrm{NS})}(\tau,u) = \int \mathcal{D}[\psi^\pm, \tilde{\psi}^\pm] e^{-S_\psi(\psi^\pm,\tilde{\psi}^\pm,a^u)} = e^{-2\pi\frac{(\mathrm{Im}\, u)^2}{\tau_2}} \frac{|\theta_3(\tau,u)|^2}{|\eta(\tau)|^2}, \tag{C.14}$$

$$Z_{\mathrm{gh}}(\tau) = \int \mathcal{D}[b,\tilde{b},c,\tilde{c}] e^{-S_{\mathrm{gh}}(b,\tilde{b},c,\tilde{c})} = \tau_2\,|\eta(\tau)|^4\,. \tag{C.15}$$

  The factor $e^{-2\pi\frac{(\mathrm{Im}\, u)^2}{\tau_2}}$ included in (C.14) is the correct anomaly factor to assure the modular invariance.

Gathering all the contributions (C.10), (C.13), (C.14) and (C.15), we finally obtain the desired partition function (2.38).

We further make a comment on the several path-integral formulas for the H_3^+ (gauged) WZW models presented in [30, 31];

$$\int \mathcal{D}g\, e^{-\kappa S_{\mathrm{WZW}}(h^u g h^{u\,\dagger})} \equiv \mathrm{Tr}\left(q^{L_0-\frac{c_g}{24}} \bar{q}^{\tilde{L}_0-\frac{c_g}{24}}\, e^{2\pi i(u j_0^3 - \bar{u}\tilde{j}_0^3)}\right) \propto \frac{e^{-(\kappa-2)\pi\frac{(\mathrm{Im}\, u)^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}, \tag{C.16}$$

$$\int \mathcal{D}g\, e^{-\kappa S^{(V)}(g,h^u,h^{u\,\dagger})} \propto \frac{e^{2\pi\frac{(\mathrm{Im}\, u)^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}, \tag{C.17}$$

$$\int \mathcal{D}g\, e^{-\kappa S^{(A)}(g,h^u,h^{u\,\dagger})} \propto \frac{e^{2\pi\frac{(\mathrm{Im}\, u)^2}{\tau_2} - \pi\kappa\frac{|u|^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}, \tag{C.18}$$

where h^u is defined in (C.5). The second and third formulas (C.17), (C.18) are modular invariant, but the first one (C.16) is not, although it has a natural interpretation as the trace in the operator calculus.

The third formula (C.18) is the easiest to prove. The axial action $S^{(A)}(g, h^u, h^{u\,\dagger})$ can be rewritten as a complete quadratic form by taking suitable coordinates on H_3^+ [31]. The relevant calculation is then reduced to successive gaussian integrals and the chiral anomaly formulas (ϕ is a non-compact real scalar along the $\mathbb{R}\sigma_2$-direction and v is a complex scalar);

$$\begin{aligned}
\int \mathcal{D}g\, e^{-\kappa S^{(A)}(g,h^u,h^{u\,\dagger})} &= \int \mathcal{D}[\phi, v, \bar{v}] \times \\
&\quad \times e^{-\frac{\kappa}{\pi}\int d^2w\{(\partial_w\phi + a^u_w)(\partial_{\bar{w}}\phi + a^u_{\bar{w}}) + (\partial_w + \partial_w\phi + a^u_w)\bar{v}(\partial_{\bar{w}} + \partial_{\bar{w}}\phi + a^u_{\bar{w}})v\}} \\
&= \int \mathcal{D}\phi\, e^{-\frac{\kappa}{\pi}\int d^2w|\partial_w\phi + a^u_w|^2} \times \\
&\quad \times \det\Big((\partial_{\bar{w}} + \partial_{\bar{w}}\phi + a^u_{\bar{w}})^\dagger(\partial_{\bar{w}} + \partial_{\bar{w}}\phi + a^u_{\bar{w}})\Big)^{-1} \\
&= \int \mathcal{D}\phi e^{-\frac{\kappa}{\pi}\int d^2w|\partial_w\phi + a^u_w|^2 + \frac{2}{\pi}\int d^2w\,|\partial_w\phi|^2 + \frac{1}{2\pi i}\int \phi R} \times \\
&\quad \times \det\Big((\partial_{\bar{w}} + a^u_{\bar{w}})^\dagger(\partial_{\bar{w}} + a^u_{\bar{w}})\Big)^{-1} \\
&\propto \frac{e^{-\pi\kappa\frac{|u|^2}{\tau_2}}}{\sqrt{\tau_2}|\eta(\tau)|^2} \times \left(\frac{e^{-2\pi\frac{(\mathrm{Im}\,u)^2}{\tau_2}}|\theta_1(\tau, u)|^2}{|\eta(\tau)|^2}\right)^{-1} \\
&= \frac{1}{\sqrt{\tau_2}}\frac{e^{2\pi\frac{(\mathrm{Im}\,u)^2}{\tau_2} - \pi\kappa\frac{|u|^2}{\tau_2}}}{|\theta_1(\tau, u)|^2}\,.
\end{aligned} \tag{C.19}$$

We have thus obtained the formula (C.18). In the last line the path-integration of ϕ is evaluated as

$$\int \mathcal{D}\phi\, e^{-\frac{\kappa-2}{\pi}\int d^2w\,\partial_w\phi\partial_{\bar{w}}\phi - \frac{\kappa}{\pi}\int d^2w\,(\partial_w\phi a^u_{\bar{w}} + \partial_{\bar{w}}\phi a^u_w) - \frac{\kappa}{\pi}\int d^2w\,|a^u_w|^2} =$$

$$e^{-\pi\kappa\frac{|u|^2}{\tau_2}}\int \mathcal{D}\phi\, e^{-\frac{\kappa-2}{\pi}\int d^2w\,\partial_w\phi\partial_{\bar{w}}\phi + \frac{i\kappa}{2\pi}\int \tilde{a}^u\wedge d\phi} \propto \frac{e^{-\pi\kappa\frac{|u|^2}{\tau_2}}}{\sqrt{\tau_2}|\eta(\tau)|^2}\,, \tag{C.20}$$

where we set $\tilde{a}^u = a^u_{\bar{w}}d\bar{w} - a^u_w dw \equiv id\Phi^u$, which satisfies

$$\oint_\alpha \tilde{a}^u = 2\pi i s_1\,, \qquad \oint_\beta \tilde{a}^u = 2\pi i s_2\,. \tag{C.21}$$

Since ϕ is non-compact, we have $\oint_\alpha d\phi = \oint_\beta d\phi = 0$, and hence the linear term of ϕ does not contribute.

The remaining formulas (C.16) and (C.17) are readily derived from (C.18) by using the properties $S_{\mathrm{WZW}}(h^u h^{u\,\dagger}) = \frac{\pi(\mathrm{Im}\,u)^2}{\tau_2}$, $S_{\mathrm{WZW}}(h^u h^{u\,\dagger\,-1}) = -\frac{\pi(\mathrm{Re}\,u)^2}{\tau_2}$.

References

[1] D. Ghoshal and C. Vafa, $c = 1$ *string as the topological theory of the conifold*, *Nucl. Phys.* **B 453** (1995) 121 [hep-th/9506122].

[2] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164].

[3] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *J. High Energy Phys.* **10** (1998) 004 [hep-th/9808149].

[4] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, *J. High Energy Phys.* **10** (1999) 035 [hep-th/9907178].

[5] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *J. High Energy Phys.* **10** (1999) 034 [hep-th/9909110]; *Comments on double scaled little string theory*, *J. High Energy Phys.* **01** (2000) 023 [hep-th/9911039].

[6] O. Pelc, *Holography, singularities on orbifolds and 4d $N = 2$ SQCD*, *J. High Energy Phys.* **03** (2000) 012 [hep-th/0001054].

[7] T. Eguchi and Y. Sugawara, *Modular invariance in superstring on Calabi-Yau N-fold with A-D-E singularity*, *Nucl. Phys.* **B 577** (2000) 3 [hep-th/0002100].

[8] S. Mizoguchi, *Modular invariant critical superstrings on four-dimensional Minkowski space x two-dimensional black hole*, *J. High Energy Phys.* **04** (2000) 014 [hep-th/0003053].

[9] S. Mizoguchi, *Noncompact gepner models for type-II strings on a conifold and an ale instanton*, hep-th/0009240.

[10] S. Yamaguchi, *Gepner-like description of a string theory on a non-compact singular Calabi-Yau manifold*, *Nucl. Phys.* **B 594** (2001) 190 [hep-th/0007069]; *Noncompact Gepner models with discrete spectra*, *Phys. Lett.* **B 509** (2001) 346 [hep-th/0102176]; *Coset character identities in superstring compactifications*, *J. High Energy Phys.* **01** (2002) 023 [hep-th/0112004].

[11] M. Naka and M. Nozaki, *Singular Calabi-Yau manifolds and ADE classification of CFTS*, *Nucl. Phys.* **B 599** (2001) 334 [hep-th/0010002].

[12] K. Hori and A. Kapustin, *Worldsheet descriptions of wrapped NS five-branes*, *J. High Energy Phys.* **11** (2002) 038 [hep-th/0203147].

[13] S. Murthy, *Notes on non-critical superstrings in various dimensions*, *J. High Energy Phys.* **11** (2003) 056 [hep-th/0305197].

[14] M. Gutperle and A. Strominger, *Timelike boundary Liouville theory*, *Phys. Rev.* **D 67** (2003) 126002 [hep-th/0301038].

[15] A. Strominger and T. Takayanagi, *Correlators in timelike bulk Liouville theory*, *Adv. Theor. Math. Phys.* **7** (2003) 369 [hep-th/0303221].

[16] V. Schomerus, *Rolling tachyons from Liouville theory*, *J. High Energy Phys.* **11** (2003) 043 [hep-th/0306026].

[17] Y. Nakayama, *Liouville field theory: a decade after the revolution*, hep-th/0402009.

[18] T. Eguchi and Y. Sugawara, *Modular bootstrap for boundary $N = 2$ Liouville theory*, *J. High Energy Phys.* **01** (2004) 025 [hep-th/0311141].

[19] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory, I. Boundary state and boundary two-point function*, hep-th/0001012.

[20] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138.

[21] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[22] C. Ahn, M. Stanishkov and M. Yamamoto, *One-point functions of $N = 2$ super-Liouville theory with boundary*, *Nucl. Phys.* **B 683** (2004) 177 [hep-th/0311169].

[23] Y. Kazama and H. Suzuki, *New $N = 2$ superconformal field theories and superstring compactification*, *Nucl. Phys.* **B 321** (1989) 232.

[24] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, unpublished.

[25] K. Hori and A. Kapustin, *Duality of the fermionic 2d black hole and $N = 2$ Liouville theory as mirror symmetry*, *J. High Energy Phys.* **08** (2001) 045 [hep-th/0104202].

[26] D. Tong, *Mirror mirror on the wall: on two-dimensional black holes and Liouville theory*, *J. High Energy Phys.* **04** (2003) 031 [hep-th/0303151].

[27] A. Hanany, N. Prezas and J. Troost, *The partition function of the two-dimensional black hole conformal field theory*, *J. High Energy Phys.* **04** (2002) 014 [hep-th/0202129].

[28] J.M. Maldacena, H. Ooguri and J. Son, *Strings in AdS_3 and the* $\mathrm{SL}(2,\mathbb{R})$ *WZW model, II. Euclidean black hole*, *J. Math. Phys.* **42** (2001) 2961 [hep-th/0005183].

[29] D. Israel, C. Kounnas and M.P. Petropoulos, *Superstrings on NS5 backgrounds, deformed AdS_3 and holography*, *J. High Energy Phys.* **10** (2003) 028 [hep-th/0306053].

[30] K. Gawędzki and A. Kupiainen, *Coset construction from functional integrals*, *Nucl. Phys.* **B 320** (1989) 625.

[31] K. Gawędzki, *Noncompact WZW conformal field theories*, hep-th/9110076.

[32] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314;
G. Mandal, A.M. Sengupta and S.R. Wadia, *Classical solutions of two-dimensional string theory*, *Mod. Phys. Lett.* **A 6** (1991) 1685;
I. Bars and D. Nemeschansky, *String propagation in backgrounds with curved space-time*, *Nucl. Phys.* **B 348** (1991) 89;
S. Elitzur, A. Forge and E. Rabinovici, *Some global aspects of string compactifications*, *Nucl. Phys.* **B 359** (1991) 581.

[33] R. Dijkgraaf, H. Verlinde and E. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B 371** (1992) 269.

[34] E. Witten, *Elliptic genera and quantum field theory*, *Commun. Math. Phys.* **109** (1987) 525.

[35] A. Polishchuk, *M.P. Appell's function and vector bundles of rank 2 on elliptic curves*, math.AG/9810084.

[36] A.M. Semikhatov, A. Taormina and I.Y. Tipunin, *Higher-level Appell functions, modular transformations, and characters*, math.QA/0311314.

[37] D. Israel, A. Pakman and J. Troost, *Extended* $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ *characters, or modular properties of a simple non-rational conformal field theory*, *J. High Energy Phys.* **04** (2004) 045 [hep-th/0402085].

[38] D. Karabali and H.J. Schnitzer, *Brst quantization of the gauged WZW action and coset conformal field theories*, *Nucl. Phys.* **B 329** (1990) 649.

[39] L.J. Dixon, M.E. Peskin and J. Lykken, *$N = 2$ superconformal symmetry and* $\mathrm{SO}(2,1)$ *current algebra*, *Nucl. Phys.* **B 325** (1989) 329.

JHEP05(2004)014

[40] B.L. Feigin, A.M. Semikhatov and I.Y. Tipunin, *Equivalence between chain categories of representations of affine* SL(2) *and* $N = 2$ *superconformal algebras*, *J. Math. Phys.* **39** (1998) 3865 [hep-th/9701043];
B.L. Feigin, A.M. Semikhatov, V.A. Sirota and I.Y. Tipunin, *Resolutions and characters of irreducible representations of the* $N = 2$ *superconformal algebra*, *Nucl. Phys.* **B 536** (1998) 617 [hep-th/9805179].

[41] W. Boucher, D. Friedan and A. Kent, *Determinant formulae and unitarity for the* $N = 2$ *superconformal algebras in two-dimensions or exact results on string compactification*, *Phys. Lett.* **B 172** (1986) 316.

[42] J.M. Maldacena and H. Ooguri, *Strings in* AdS_3 *and* $\mathrm{SL}(2,\mathbb{R})$ *WZW model, I*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053].

[43] J.M. Evans, M.R. Gaberdiel and M.J. Perry, *The no-ghost theorem for* AdS_3 *and the stringy exclusion principle*, *Nucl. Phys.* **B 535** (1998) 152 [hep-th/9806024].

[44] J. Balog, L. O'Raifeartaigh, P. Forgacs and A. Wipf, *Consistency of string propagation on curved space-times: an* SU(1,1) *based counterexample*, *Nucl. Phys.* **B 325** (1989) 225;
P.M.S. Petropoulos, *Comments on* SU(1,1) *string theory*, *Phys. Lett.* **B 236** (1990) 151;
N. Mohammedi, *Int. J. Mod. Phys.* **A 5** (1990) 3201;
I. Bars and D. Nemeschansky, *String propagation in backgrounds with curved space-time*, *Nucl. Phys.* **B 348** (1991) 89;
S. Hwang, *No ghost theorem for* SU(1,1) *string theories*, *Nucl. Phys.* **B 354** (1991) 100.

[45] A. Pakman, *Unitarity of supersymmetric* $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ *and no-ghost theorem for fermionic strings in* $AdS_3 \times N$, *J. High Energy Phys.* **01** (2003) 077 [hep-th/0301110].

[46] S. Ribault and V. Schomerus, *Branes in the 2-d black hole*, *J. High Energy Phys.* **02** (2004) 019 [hep-th/0310024].

[47] D. Gepner, *Exactly solvable string compactifications on manifolds of* $\mathrm{SU}(N)$ *holonomy*, *Phys. Lett.* **B 199** (1987) 380; *Space-time supersymmetry in compactified string theory and superconformal models*, *Nucl. Phys.* **B 296** (1988) 757.

[48] D. Gepner and Z.-a. Qiu, *Modular invariant partition functions for parafermionic field theories*, *Nucl. Phys.* **B 285** (1987) 423.

[49] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with* $\mathrm{SU}(N)$ *holonomy*, *Nucl. Phys.* **B 315** (1989) 193.

[50] Y. Hikida and Y. Sugawara, *Superstring vacua of 4-dimensional pp-waves with enhanced supersymmetry*, *J. High Energy Phys.* **10** (2002) 067 [hep-th/0207124].

[51] E. Witten, *On the Landau-Ginzburg description of* $N = 2$ *minimal models*, *Int. J. Mod. Phys.* **A 9** (1994) 4783 [hep-th/9304026].

[52] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Birkhäuser, 1985.

[53] T. Kawai, Y. Yamada and S.-K. Yang, *Elliptic genera and* $N = 2$ *superconformal field theory*, *Nucl. Phys.* **B 414** (1994) 191 [hep-th/9306096].

[54] S. Odake, *Extension of* $N = 2$ *superconformal algebra and Calabi-Yau compactification*, *Mod. Phys. Lett.* **A 4** (1989) 557; *Character formulas of an extended superconformal algebra relevant to string compactification*, *Int. J. Mod. Phys.* **A 5** (1990) 897.

JHEP05(2004)014

[55] C. Itzykson and J.-M. Drouffe, *Statistical field theory*, Cambridge University Press, Cambridge 1989, vol 2, appendix 9.A.

[56] T. Eguchi and A. Taormina, *Character formulas for the $N = 4$ superconformal algebra*, *Phys. Lett.* **B 200** (1988) 315; *On the unitary representations of $N = 2$ and $N = 4$ superconformal algebras*, *Phys. Lett.* **B 210** (1988) 125.

[57] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, *Nucl. Phys.* **B 584** (2000) 69 [hep-th/9906070]; erratum *ibid.* **B 608** (2001) 477.

[58] A.D. Shapere and C. Vafa, *BPS structure of Argyres-Douglas superconformal theories*, hep-th/9910182.

[59] K. Miki, *The representation theory of the* SO(3) *invariant superconformal algebra*, *Int. J. Mod. Phys.* **A 5** (1990) 1293.

[60] V.K. Dobrev, *Characters of the unitarizable highest weight modules over the $N = 2$ superconformal algebras*, *Phys. Lett.* **B 186** (1987) 43;
E. Kiritsis, *Character formulae and the structure of the representations of the $N = 1$, $N = 2$ superconformal algebras*, *Int. J. Mod. Phys.* **A 3** (1988) 1871.